Future topics

1. (2006 Nov 27) Find all polynomials $f$ such that $f(2t)$ can be written as a polynomial in $f(t)$, i.e., for which there exists a polynomial $h$ such that

$$f(2t) = h(f(t)) \,.$$

Use the identity $\sin^2 2t = 4\sin^2 t(1-\sin^2 t)$ to show that $\sin t$ is not a polynomial.[1]

2. For a polynomial $p$ and a natural number $k$, let $[t^k]p(t)$ denote the coefficient of $t^k$ in $p(t)$. (If $k > \deg p$, then, naturally, $[t^k]p(t) = 0$.) In our meeting of January 24 we observed that, since $\mathbb{Z}$ is a ring, we obviously have

$$(\forall k \in \mathbb{N}\colon [t^k]p(t) \in \mathbb{Z}) \implies (\forall n \in \mathbb{Z}\colon p(n) \in \mathbb{Z}) \,.$$

In words: if all a polynomial's coefficients are integers, then it takes integer values for integer arguments. We also noted that the converse is false, and gave some examples. New question: Is there any ring for which the converse is true? (See our notes for August 22 for a summary of what we know so far.)

3. Show that, if $x, y, z$ are nonnegative real numbers, then[2]

$$\frac{3\sqrt{3}}{2\sqrt{x+y+z}} \le \frac{\sqrt{x}}{y+z} + \frac{\sqrt{y}}{z+x} + \frac{\sqrt{z}}{x+y} \,.$$

4. Prove[3] that $(4\cos^2 9^\circ - 3)(4\cos^2 27^\circ - 3) = \tan 9^\circ$.

5. A problem from Levin: Given a sequence $(a_n)_{n=1}^\infty$ of positive real numbers, does there necessarily exist a sequence $(b_n)_{n=1}^\infty$ of positive real numbers such that $\sum_{n=1}^\infty b_n = 1$ and $\sum_{n=1}^\infty a_n b_n < \infty$?

6. Recall that, given vector spaces $U$ and $V$ (over some field $F$), we say that a function $f\colon U \to V$ is *linear* if

$$f(x+y) = f(x) + f(y) \tag{1}$$
$$\text{and} \qquad f(\alpha x) = \alpha f(x) \tag{2}$$

for all $x, y \in U$ and all $\alpha \in F$. Are the conditions (1) and (2) independent? That is, is it possible for a function $f$ to satisfy one but not the other?

[1] Exercise 1.1.13 from *Polynomials*, by E.J. Barbeau.

[2] This problem was suggested to me by Dr. Byron Schmuland; it's based on an inequality in J. Michael Steele, *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities* (Cambridge UP, 2004), 131.

[3] Titu Andreescu and Zuming Feng, *103 Trigonometry Problems* (Boston: Birkhäuser, 2005), introductory problem #16.