Math Club Notes: 2007 February 16

We spent most of this meeting drinking, but we did have a look at this problem from our list:

> Let $\mathcal{L}$ be a topological vector space. (That's a vector space with a Hausdorff topology such that the vector operations are continuous.) I found the following theorem in a book: For every $U \subset \mathcal{L}$ which is a neighbourhood of $\vec{0}$, and every $\vec{x} \in \mathcal{L}$, there exist $\alpha \in \mathbb{R}$ and $\vec{y} \in U$ such that $\vec{x} = \alpha\vec{y}$. The author's proof: This is an immediate consequence of the continuity of $\alpha \mapsto \alpha\vec{x}$ at the point $\alpha = 0$. (I have rephrased both theorem and proof.) Questions: What proof did the author have in mind, and what's wrong with it?

The second question first: there's nothing wrong with the author's proof. (I thought I had a counterexample using $\mathbb{R}^n$ with the discrete topology, but I was getting confused about the direction of my continuous functions.)

The author tells us we should use the continuity of the function $\alpha \mapsto \alpha\vec{x}$ at $\alpha = 0$. That this function is continuous is part of the definition of a topological vector space: scalar multiplication is continuous (in both variables, but all we care about here is continuity in the scalar variable).

Now, if we were dealing with $\mathbb{R}^n$ (or indeed, any normed linear space), we might want to unpack the definition of continuity in terms of ϵ and δ. But since we're dealing with an arbitrary topological linear space, we can't assume the existence of a norm, so the ϵ-δ definition doesn't make sense. In this context, the appropriate definition of continuity is that a function is continuous if the preimages of open sets under that function are open.

So the fact we wish to use is: if G is an open set in $\mathcal{L}$, then its preimage is an open set in $\mathbb{R}$. Well, we have an open set, namely U, so we naturally consider its preimage $A = \{\alpha\colon \alpha\vec{x} \in U\}$. The author also suggested we look specifically at $\alpha = 0$, and indeed, for this value we find that $\alpha\vec{x} = \vec{0} \in U$, since U is a neighbourhood of $\vec{0}$; thus $0 \in A$.

So A is an open set in $\mathbb{R}$ and contains 0.

Now, what we wish to find is $\alpha \in \mathbb{R}$ and $\vec{y} \in U$ such that $\vec{x} = \alpha\vec{y}$. We seek $\vec{y} \in U$, and the only vectors we know to be in U are those of the form $\alpha\vec{x}$ where $\alpha \in A$; so we will choose some $\alpha \in A$ and set $\vec{y} = \alpha\vec{x}$. Evidently this α is not the same α as in the statement of the question, since it's on the wrong side of the equality. So let's call it β instead.

What we have so far is a partial solution that looks like this:

> Let $A = \{\alpha\colon \alpha\vec{x} \in U\}$. Since A is the preimage of the open set U under the continuous function $\alpha \mapsto \alpha\vec{x}$, A is open. Since $0\vec{x} = \vec{0} \in U$ by hypothesis, $0 \in A$.

Choose $\beta \in A$ such that… [we don't know yet]. Let $\vec{y} = \beta\vec{x}$. Then $\vec{y} \in U$ by the choice of β.

What else do we need? That, for some α, $\vec{x} = \alpha\vec{y}$. So we take $\alpha = \frac{1}{\beta}$, which tells us we'd better have $\beta \neq 0$. Making these additions yields this partial solution:

> Let $A = \{\alpha\colon \alpha\vec{x} \in U\}$. Since A is the preimage of the open set U under the continuous function $\alpha \mapsto \alpha\vec{x}$, A is open. Since $0\vec{x} = \vec{0} \in U$ by hypothesis, $0 \in A$.
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> Choose $\beta \in A$ such that $\beta \neq 0$ and … [we don't know what else]. Let $\vec{y} = \beta\vec{x}$. Then $\vec{y} \in U$, as desired, since $\beta \in A$, and since $\beta \neq 0$ we can let $\alpha = \frac{1}{\beta}$ and have $\vec{x} = \alpha\vec{y}$, also as desired.

Now that we have constructed α and $\vec{y}$ as desired, we see that once we have $\beta \in A$ and $\beta \neq 0$, we need nothing else.

The only thing that remains to be explained is how we know such β exists. Here it is crucial that A is an open set in $\mathbb{R}$, in the usual topology of $\mathbb{R}$. For in that topology, singleton sets are not open; thus it is impossible that $A = \{0\}$, and so A contains a nonzero point, which we can take as β. Thus:

> Let $A = \{\alpha\colon \alpha\vec{x} \in U\}$. Since A is the preimage of the open set U under the continuous function $\alpha \mapsto \alpha\vec{x}$, A is open. Since $0\vec{x} = \vec{0} \in U$ by hypothesis, $0 \in A$.
>
> Choose $\beta \in A$ such that $\beta \neq 0$. (This is possible because A is an open subset of $\mathbb{R}$ and so is not a singleton set.) Let $\vec{y} = \beta\vec{x}$. Then $\vec{y} \in U$, as desired, since $\beta \in A$, and since $\beta \neq 0$ we can let $\alpha = \frac{1}{\beta}$ and have $\vec{x} = \alpha\vec{y}$, also as desired.

And that's it.

(By the way, what use does this solution make of the fact that $0 \in A$?)