Math Club Notes: 2007 January 24

1 Matrix sum equal to matrix product

One of our outstanding problems: for which pairs of matrices A and B do we have $A + B = AB$?

Since

$$AB - (A + B) = (A - I)(B - I) - I\,,$$

we have

$$A + B = AB \iff (A - I)(B - I) = I\,.$$

So the desired pairs of matrices are those of the form

$$A = P + I \qquad \text{and} \qquad B = P^{-1} + I$$

for an invertible matrix P. For example, with $P = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ we obtain

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\,.$$

Note that this solution works just as well in any ring with unity. For example, in $\mathbb{Z}$, the invertible elements are ± 1, giving rise to solutions $A = B = 0$ and $A = B = 2$. (In fact, the problem is much easier when it is stated as a problem about rings, since then you don't get distracted by all your other knowledge of matrices.)

The trick in this solution is the same as that used in the problem about rectangles from our meeting on 2005 August 8.

2 Expressing open sets as preimages of open sets

Another from our list of problems: Can every open set in $\mathbb{R}^n$ be expressed as the continuous preimage of an open set in $\mathbb{R}$? That is, given an open set $G \subset \mathbb{R}^n$, does there necessarily exist a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ and an open set $H \subset \mathbb{R}$ such that $G = f^{-1}(H)$?

(This question is about the converse of the fact we know from topology that the continuous preimage of an open set is open.)

Turns out the answer is yes. In fact, we can always take $H = (0, \infty)$ and choose f so that $G = f^{-1}(H)$. In such a case, what we want is f such that

$$\begin{cases} f(x) > 0 & \text{if } x \in G, \\ f(x) = 0 & \text{if } x \notin G. \end{cases}$$

It's easy enough to arrange that $f(x) = 0$ for $x \notin G$. But what value shall we assign for $x \in G$?

Well, what do we know about $x \in G$? Since G is open, there is an open ball of positive radius centred at x and contained in G. Positive radius — that sounds just like what we want.

Of course, we can't just say "for $x \in G$, let $f(x)$ be that value such that an open ball centred at x and of radius $f(x)$ is contained in G", since there are many such values. But what we can do is take the largest. Or rather, since there might not be a largest,[1] the supremum. Thus we get a candidate f:

$$f(x) = \begin{cases} \sup\{\delta\colon B(x,\delta) \subset G\} & \text{if } x \in G, \\ 0 & \text{if } x \notin G. \end{cases}$$

Another way to express this function:

$$f(x) = \inf\{|x - y|\colon y \in G^C\}\,.$$

In other words, $f(x) = \operatorname{dist}(x, G^C)$.

Proving that this f has the desired properties is left as an exercise. (There is one special case that should be handled separately, namely the case $G = \mathbb{R}^n$.)

Cindy noted a similarity between this problem and a theorem mentioned in our real analysis course last term: given two disjoint compact sets $K, L \subset \mathbb{R}^n$, there exists a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = 0$ for all $x \in K$ and $f(x) = 1$ for all $x \in L$. Proving this theorem makes a good follow-up exercise.

Our solution here crucially relies on the fact that $\mathbb{R}^n$ has a distance function, so it cannot be extended to arbitrary topological spaces. Not knowing much about topology, I don't know whether there are topological spaces with open sets that cannot be expressed as the preimages of open sets in $\mathbb{R}$. (I'd be pretty surprised if there aren't, though.)

3 Polynomials with integer values for integer arguments

From our list: It is easy to see that if all of a polynomial p's coefficients are integers, then it takes integer values for integer arguments (i.e., $n \in \mathbb{Z} \Rightarrow p(n) \in \mathbb{Z}$). Is the converse true? That is: Suppose p is a polynomial such that, for every integer n, $p(n)$ is an integer. Is it necessarily true that all of p's coefficients are integers?

The converse is false. The simplest counterexample is

$$p(t) = \frac{1}{2}t^2 + \frac{1}{2}t = \frac{t(t+1)}{2}\,.$$

If $t \in \mathbb{Z}$, then one of t and $t+1$ is even, so $p(t) \in \mathbb{Z}$; but some of its coefficients are not integers.

[1] This qualification is habitual. Is it needed here?

This example can be extended to produce such polynomials with any degree ≥ 2, e.g.,

$$p(t) = \frac{t^2(t+1)}{2} .$$

The converse does, however, hold for linear and constant polynomials. I'll leave proving that as an exercise. It can also be shown that such a polynomial must have rational coefficients.

The example can also be extended to produce polynomials with any desired positive integer in the denominator of their non-integer coefficients, e.g.,

$$p(t) = \frac{t(t+1)(t+2)(t+3)(t+4)}{5} .$$

In our meeting I mentioned that we can think of binomial coefficients as polynomials, defining

$$\binom{t}{k} = \frac{t(t-1)(t-2)\cdots(t-k+1)}{k!} ,$$

which is a polynomial in t of degree k. Such polynomials, I claimed, take integer values at integer arguments, and we know this combinatorially: $\binom{n}{k}$ is the number of k-element subsets of a set of n objects, and hence is an integer. This argument is incomplete; we might revisit this topic next meeting.

4 Erroneous argument involving the tangent of a sum

From our list: Consider the following argument, similar to one given in the notes for January 10. Let $a \in \mathbb{R}$ be such that $\tan a \neq 0$. Let $b = \arctan \frac{1}{\tan a}$. Then $\tan b = \frac{1}{\tan a}$, and so

$$\tan a \tan b = 1 ,$$

so that $1 - \tan a \tan b = 0$, whence (by the addition formula for tangent),

$$\tan a + \tan b = (1 - \tan a \tan b)\tan(a+b) = 0 .$$

But wait! If $\tan a + \tan b = 0$, then $\tan a$ and $\tan b$ must be of opposite signs. But by construction their product is 1, which is positive, so they have the same sign. What happened?

Radoslav wondered where that addition formula came from. It's more normally written in the form

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

and is easily proven from the addition formulæ for sin and cos. The problem is that, in that proof, we divide by $\cos a$, by $\cos b$, and by $\cos(a+b)$, so the result is not valid if any of these are zero, that is, if any of a, b, and $a + b$ are of the form $(n + \frac{1}{2})\pi$ (where $n \in \mathbb{Z}$).

The construction of b given above guarantees that $a + b$ is of this form.

In the similar argument given in our meeting of January 10, this problem arises if the triangle in question is right; otherwise all is well.