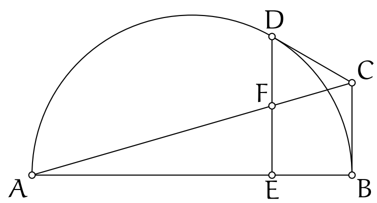
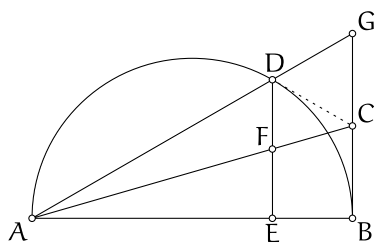
Math Club Notes: 2007 January 10

1 Archimedes' weird bisection lemma

One of our long-outstanding problems has been to prove Proposition 2 from Archimedes' *Book of Lemmas*: Let AB be the diameter of a semicircle, and let the tangents to it at B and at any other point D on it meet in C. If now DE be drawn perpendicular to AB, and if AC, DE meet in F, then $DF = FE$.

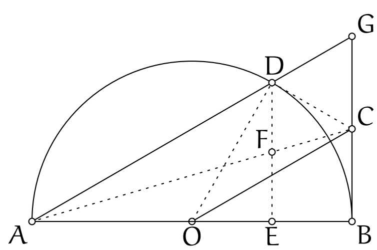


I like to look at the bisection of DE as resulting from two "projection" operations. First, join AD and extend it to meet BC (extended) at G.



Since $DE \perp AB$ and $BC \perp AB$ by construction, $DE \parallel BC$. Therefore they are cut in proportion by the pencil of lines through A; we have $EF : FD = BC : CG$.

Now, let the centre of the circle be O; join OC and OD.



Since CB and CD are tangents from the same external point, $\triangle CBO = \triangle CDO$, whence $\angle BOC = \frac{1}{2}\angle BOD$. Since $\angle BOD$ is the central angle on the same arc as the inscribed angle $\angle BAD$, we have $\angle BAD = \frac{1}{2}\angle BOD$. Thus $\angle BAD = \angle BOC$, and so $AG \parallel OC$. Thus these lines cut off proportional segments on their transversals; we have $BC : CG = BO : OA$.

Certainly O bisects AB, so we're done.

2 Area of an average rectangle

Another of our outstanding problems: Given n rectangles, each of area at least 1. Let R be the rectangle whose width is the average of the given rectangles' widths and whose height is the average of the given rectangles' heights. Prove that R has area at least 1.

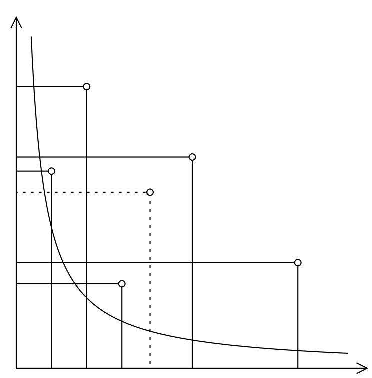
To prove this algebraically, we use the Cauchy-Schwarz inequality:

$$\Big(\sum_{k=1}^{n} a_k^2\Big)\Big(\sum_{k=1}^{n} b_k^2\Big) \geq \Big(\sum_{k=1}^{n} a_k b_k\Big)^2 . \tag{1}$$

For our problem, let the kth rectangle have width x_k and height y_k. We are given that $x_k y_k \geq 1$ for all k. The area of the average rectangle is

$$\begin{aligned}
&\Big(\sum_{k=1}^{n} \frac{x_k}{n}\Big)\Big(\sum_{k=1}^{n} \frac{y_k}{n}\Big) \\
&\geq \Big(\sum_{k=1}^{n} \frac{\sqrt{x_k y_k}}{n}\Big)^2 && \text{(by (1), with } a_k, b_k := \sqrt{x_k/n}, \sqrt{y_k/n}\text{)} \\
&\geq \Big(\sum_{k=1}^{n} \frac{1}{n}\Big)^2 && (x_k y_k \geq 1 \text{ by hypothesis}) \\
&= 1 .
\end{aligned}$$

More geometrically, we can proceed by placing all the rectangles on the Cartesian plane, with their sides parallel to the axes and their bottom-left corners at the origin. Then the kth rectangle's upper-right corner is at (x_k, y_k); the fact that $x_k y_k \geq 1$ means, then, that all these corners lie on or above the curve $xy = 1$ (more specifically, the portion of it in the first quadrant).



The average rectangle's upper-right corner is at the centre of gravity of the given rectangles' upper-right corners; since the region in question is convex, the centre of gravity lies within it too, and so the average rectangle's area is at least 1.

3 Tangents of a triangle's angles

Yet another old problem: show that, if a, b, and c are the angles of a triangle, then

$$\tan a + \tan b + \tan c = \tan a \tan b \tan c\ . \tag{2}$$

(I got this problem from Eli Maor's *Trigonometric Delights*.)

First recall (or derive afresh) the addition formula for the tangent function:

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}\ ,$$

or, more conveniently for us,

$$\tan u + \tan v = (1 - \tan u \tan v)\tan(u+v)\ .$$

Now we compute as follows:

$$\begin{aligned}
&\tan a + \tan b + \tan c - \tan a \tan b \tan c \\
&= (1 - \tan a \tan b)\tan(a+b) + \tan c - \tan a \tan b \tan c \\
&= (1 - \tan a \tan b)(\tan(a+b) + \tan c) \\
&= (1 - \tan a \tan b)(1 - \tan(a+b)\tan c)\tan(a+b+c)
\end{aligned}$$

Now, since a, b, and c are the angles of a triangle, $\tan(a+b+c) = \tan 180^\circ = 0$, which establishes (2).

A few remarks:

1. In the computation above, we broke the symmetry between the three angles by combining $\tan a$ and $\tan b$ first. A symmetrical version of the identity we needed:

   $$\tan(a+b+c) = \frac{\tan a + \tan b + \tan c - \tan a \tan b \tan c}{1 - \tan a \tan b - \tan b \tan c - \tan c \tan a}\ .$$

2. The original identity is slightly wrong, or needs some careful interpretation: consider the case of a right-angled triangle.

3. Consider the following argument, similar to the one given above. Let $a \in \mathbb{R}$ be such that $\tan a \neq 0$. Let $b = \arctan \frac{1}{\tan a}$. Then $\tan b = \frac{1}{\tan a}$, and so

   $$\tan a \tan b = 1\ ,$$

   so that $1 - \tan a \tan b = 0$, whence

   $$\tan a + \tan b = (1 - \tan a \tan b)\tan(a+b) = 0\ .$$

   But wait! If $\tan a + \tan b = 0$, then $\tan a$ and $\tan b$ must be of opposite signs. But by construction their product is 1, which is positive, so they have the same sign. What happened?

4 Dodgson's sums of squares

Another old problem, this one from *Pillow Problems* by Charles L. Dodgson (aka Lewis Carroll), originally published 1895 by Macmillan, republished 1958 by Dover: "Prove that 3 times the sum of 3 squares is also the sum of 4 squares."

The proof is just this identity:

$$3(a^2 + b^2 + c^2) = (a + b + c)^2 + (a - b)^2 + (b - c)^2 + (c - a)^2 ,$$

which is easily verified. (The trick, of course, is coming up with it. Alas, I don't have much to say about how one does that.)

5 Reasons some functions aren't polynomials

Another problem from our list: How do we know $\sqrt{t}$ isn't a polynomial? What about logarithms? Exponentials?

There are several reasons for each type of function. In each case, we know the Taylor series, and it isn't finite. (In the case of square roots, we need to expand around, say, 1.) All these functions have infinitely many nonzero derivatives.

In the case of $\sqrt{t}$, we should clarify a bit, since $\sqrt{t}$ is only defined for $t \geq 0$ but polynomials are defined on all of $\mathbb{R}$. So really we are asking: is there a polynomial which agrees with $\sqrt{t}$ for nonnegative t? In other words, does $\sqrt{t}$ have a polynomial extension?

Another way to see that it doesn't: the first derivative (from the right) of $\sqrt{t}$ at $t = 0$ is ∞. But the derivative of a polynomial is a polynomial, hence finite-valued everywhere.

Another: if $\sqrt{t}$ is a polynomial, it has some degree. Since $\sqrt{t}$ isn't constant, $\deg \sqrt{t} \geq 1$. But then, composing it with t^2, we obtain

$$(\sqrt{t})^2 = t ,$$

which by taking degrees yields

$$(\deg t^2)(\deg \sqrt{t}) = \deg t ,$$

and so $\deg \sqrt{t} = \frac{1}{2}$. But polynomials have integer degrees.

Another, pointed out by Vish: if $\sqrt{t}$ is a polynomial, then, composing it the *other* way with t^2, we learn that

$$\sqrt{t^2} = |t|$$

is a polynomial. But it isn't. ($|t|$ is different from t, but agrees with it at infinitely many values; this can't happen with polynomials.)

As for the logarithm function, we could note that it has an asymptote. But even easier is to recall that

$$\log(t^2) = 2\log t$$

and take degrees. A similar argument works for the exponential function.