Math Club Notes: 2006 November 27

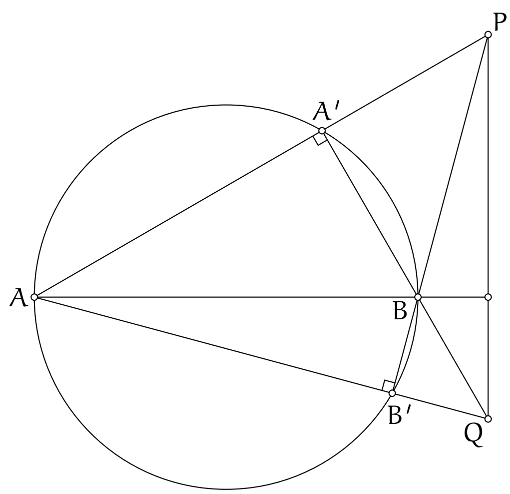
1 A construction with straightedge alone

A problem I spotted on someone's Math 241 homework (paraphrased):

> Given a circle, its diameter AB, and a point P lying neither on the circle nor on the given diameter extended, construct with straightedge alone a line passing through P and perpendicular to AB (extended).

(The traditional solution using both compass and straightedge is Euclid I.12.)

Since the only thing a straightedge can do is join points to form lines (and thereby find new points as intersections), there's only so much you can try; so it's not hard to find the right construction.



Construction Join PA, intersecting the circle also at A'. Join PB, intersecting the circle also at B'. Join $A'B$ and AB', and extend them to intersect at Q. Join PQ. Then $PQ \perp AB$, as desired.

Proof AB is a diameter, so $\angle AA'B$ and $\angle AB'B$ are right angles (Euclid III.31). Thus the segments QA' and PB' are altitudes in $\triangle APQ$, and so their point of intersection, B, is the orthocentre. Therefore AB extended is the third altitude, and as such forms a right angle with PQ, as claimed. □

(This construction ignores some special cases. Identifying them and patching things up is left as an exercise.)

2 Why sine isn't a polynomial

A few reasons why sine isn't a polynomial: it has infinitely many zeroes, but isn't the zero polynomial; its Maclaurin series isn't finite; it has infinitely many nonzero derivatives; it is bounded but not constant. Finally, a nifty one, exercise 1.1.13 from *Polynomials*, by E.J. Barbeau:

> Find all polynomials f such that $f(2t)$ can be written as a polynomial in $f(t)$, i.e., for which there exists a polynomial h such that
>
> $$f(2t) = h(f(t)) \ .$$
>
> Use the identity $\sin^2 2t = 4\sin^2 t(1-\sin^2 t)$ to show that $\sin t$ is not a polynomial.

This one will go on the list of outstanding problems.

3 Making or

Last time we discussed our outstanding problem of showing that it's impossible to express $\vee$ using only $\neg$ and $\veebar$. The notes for 2006 Nov 20 include a solution for the similar problem of showing it's impossible to express $\neg$ using only $\wedge$, $\vee$, and $\veebar$; we defined the set of boolean functions expressible with these operators recursively, then showed by structural induction that they all "fix zero", and observed that negation doesn't.

To find a similar proof for the problem of expressing $\vee$, we listed all sixteen boolean functions of two boolean variables A and B, and tried to express them all using $\neg$ and $\veebar$.

0	0	1	1	A
0	1	0	1	B
0	0	0	0	$A \veebar A$
0	0	0	1	
0	0	1	0	
0	0	1	1	A
0	1	0	0	
0	1	0	1	B
0	1	1	0	$A \veebar B$
0	1	1	1	

0	0	1	1	A
0	1	0	1	B
1	0	0	0	
1	0	0	1	$\neg(A \veebar B)$
1	0	1	0	$\neg B$
1	0	1	1	
1	1	0	0	$\neg A$
1	1	0	1	
1	1	1	0	
1	1	1	1	$\neg(A \veebar A)$

It's pretty easy to find the above eight expressions. After fiddling for a while, it's easy to become convinced that they're the only ones that can be expressed using just $\neg$ and $\veebar$. (Other candidate expressions, such as $(\neg A) \veebar B$, turn out to be equivalent to ones already listed.)

After staring at the table for a while, we notice what distinguishes those eight expressions from the other eight — the expressible functions have an even number of 1s in their rows.

The simplest way to express this condition formally is:

$$f(0,0) \veebar f(0,1) \veebar f(1,0) \veebar f(1,1) = 0 \,. \tag{$*$}$$

(Note that $\veebar$ is associative, so we don't need parentheses here.) So, to show that $\vee$ cannot be expressed using only $\neg$ and $\veebar$, we must show four things:

1. The functions $(A, B) \mapsto A$ and $(A, B) \mapsto B$ have property ($*$).
2. If f has property ($*$), then $\neg f$ has it too.
3. If f and g have property ($*$), then $f \veebar g$ has it too.
4. The function $(A, B) \mapsto A \vee B$ does not have property ($*$).

I leave the details as an exercise.