Math Club Notes: 2006 October 30

1 The mediant inequality

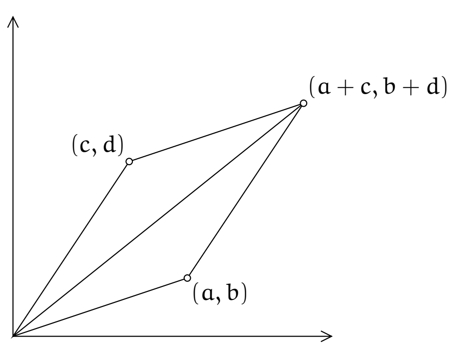
Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, the fraction obtained by adding the numerators and denominators, that is, $\frac{a+c}{b+d}$, is called their *mediant*. (The value of the mediant depends, of course, on how the fractions have been written; for example, the mediant of $\frac{1}{2}$ and $\frac{1}{1}$ is $\frac{2}{3}$, while the mediant of $\frac{2}{4}$ and $\frac{3}{3}$ is $\frac{5}{7}$. In other words, the mediant is not a function.)

A nifty fact: the mediant of two fractions lies between them. More precisely: if b and d are positive, then

$$\frac{a}{b} \le \frac{c}{d} \implies \frac{a}{b} \le \frac{a+c}{b+d} \le \frac{c}{d} \, .$$

I will call this result the *mediant inequality*. (I don't know if it has an established name.)

I'll leave the algebraic proof as an exercise, along with the investigation of what happens when the denominators are negative. (Or zero...) We can interpret the inequality geometrically by interpreting the fractions as the slopes of vectors:



The inequality asserts the intuitively obvious ordering of slopes in this figure.

1.1 Application

Today we looked at the following problem, which occurred in a U of Waterloo contest:

> Let f be a real-valued function such that
>
> 1. f is increasing on $[0, 1]$,
> 2. $f(0) = 0$, and

3. f' exists and is increasing on $(0, 1)$.

Show that $g(x) = f(x)/x$ is increasing on $(0, 1)$.

The problem is wrong; so we first found a counterexample. The key observation is that, while the existence of f' on $(0, 1)$ entails that f is continuous on $(0, 1)$, nothing in the given conditions requires that f be continuous (from the right) at 0. Thus, for example, we can take

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x \le 1. \end{cases}$$

It is easy to verify that f satisfies the three given conditions, but $g(x) = \frac{1}{x}$ is not increasing. (Finding a counterexample in which f and f' are *strictly* increasing on their respective intervals is left as an exercise.)

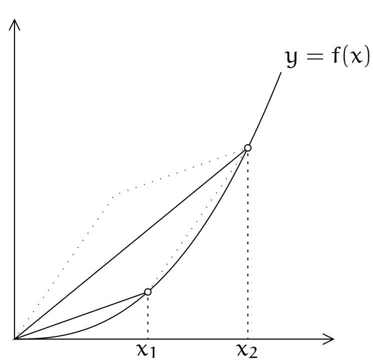
What the authors of the problem intended to ask was this:

> Let f be a real-valued function such that
>
> 1. f is *continuous* on $[0, 1]$,
> 2. $f(0) = 0$, and
> 3. f' exists and is increasing on $(0, 1)$.
>
> Show that $g(x) = f(x)/x$ is increasing on $(0, 1)$.

To solve this problem, we must notice that, since $f(0) = 0$,

$$g(x) = \frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0}$$

is the slope of the secant from $(0, 0)$ to $(x, f(x))$. So we draw a picture such as the following:



Since f' is increasing, we draw a concave-up curve. To show that g is increasing, we choose x_1, x_2 such that $0 < x_1 \le x_2 < 1$, and wish to show that $g(x_1) \le g(x_2)$. In terms of the figure, we wish to show that the secant over $[0, x_1]$ is shallower than the secant over $[0, x_2]$. Once we draw in the rest of the parallelogram, we see an opportunity to use the mediant inequality; thus we turn our attention to showing that the secant over $[0, x_1]$ is shallower than the other side of the parallelogram, that is, the secant over $[x_1, x_2]$.

What we want is information about secant slopes, and what we have (in the condition that f' be increasing) is information about tangent slopes. Cue the mean value theorem.

The complete solution is as follows.

Let f be as described. Let x_1, x_2 be such that $0 < x_1 \leq x_2 < 1$; we wish to show that $g(x_1) \leq g(x_2)$. Since f is continuous on $[0, x_1]$ and differentiable on $(0, x_1)$, by the mean value theorem there exists c_1 such that

$$0 < c_1 < x_1 \qquad \text{and} \qquad f'(c_1) = \frac{f(x_1) - f(0)}{x_1 - 0} = \frac{f(x_1)}{x_1} .$$

By the same argument for the interval $[x_1, x_2]$, there exists c_2 such that

$$x_1 < c_2 < x_2 \qquad \text{and} \qquad f'(c_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} .$$

Now, $c_1 < x_1 < c_2$ and f' is increasing, so $f'(c_1) \leq f'(c_2)$, that is,

$$\frac{f(x_1)}{x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} .$$

By what we're calling the mediant inequality,

$$\frac{f(x_1)}{x_1} \leq \frac{f(x_2)}{x_2} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} .$$

The first inequality here asserts that $g(x_1) \leq g(x_2)$, which is what we wished to prove.