Math Club Notes: 2006 May 23

1 Boolean rings

Way back on 2005 May 2, at our first meeting ever, I mentioned a few examples of "Boolean rings", and suggested that the examples make sense of the name. A much better summary can be found here: http://www.math.niu.edu/~rusin/known-math/99/boolean_ring

2 Catalan's identity

Back on 2006 Feb 9, we proved that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{3}{4}$$

by induction on n. Another approach is to interpret this sum as a Riemann sum.

Consider the function $f(x) = 1/x$ over the the interval $[n, 2n]$. Partition that interval into subintervals by breaking it up at integer values; each interval is then of width 1, so the Riemann sum obtained by evaluating f at the right endpoints of each interval is just the sum in question. Since f is strictly decreasing, this sum is a strict underestimate of the true area under the curve; thus

$$\begin{aligned}\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} &< \int_n^{2n} \frac{1}{x}\,dx \\ &= \ln|x|\Big|_n^{2n} \\ &= \ln 2n - \ln n \\ &= \ln \frac{2n}{n} \\ &= \ln 2\end{aligned}$$

So if we happen to know that $\ln 2 < \frac{3}{4}$ (indeed, $\ln 2 \approx 0.69$), then we could solve the problem this way instead.

Of course, one might wonder how we would know that $\ln 2 < \frac{3}{4}$. One way is to consider the sum

$$\ln 2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

(This can be shown by deriving the Taylor series for the function $\ln(1+x)$ about $x = 0$, then taking $x = 1$. See also our notes for June 13.) This is an alternating sum whose terms are strictly decreasing (in absolute value); thus its partial sums are bounds for its value:

$$\frac{1}{1} - \frac{1}{2} < \ln 2 < \frac{1}{1}$$
$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} < \ln 2 < \frac{1}{1} - \frac{1}{2} + \frac{1}{3}$$

and so on. By continuing in this manner, we can get arbitrarily good bounds on $\ln 2$; eventually we will find out that $\ln 2 < \frac{3}{4}$.

The connection between these sums is expressed directly in Catalan's identity:[1]

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} + \cdots - \frac{1}{2n} .$$

Vish and I worked out a proof of this identity by considering how to pair up the terms on the left and right in the special case $n = 4$:

LHS:	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$				
RHS:	$\frac{1}{1}$	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$	$-\frac{1}{6}$	$\frac{1}{7}$	$-\frac{1}{8}$

The simplest idea is to pair each term on the left with one on the right; but this can't work, since there are only n terms on the left, but $2n$ on the right. Still, the most natural thing to do for now is to pair them up by denominator:

LHS:					$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
RHS:	$\frac{1}{1}$	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$	$-\frac{1}{6}$	$\frac{1}{7}$	$-\frac{1}{8}$

Now let's introduce the missing terms on the left (and subtract them back out again):

LHS:	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
	$-\frac{1}{1}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$				
RHS:	$\frac{1}{1}$	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$	$-\frac{1}{6}$	$\frac{1}{7}$	$-\frac{1}{8}$

Now what about the signs on the right? Let's split up the positive terms and negative terms:

LHS:	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
	$-\frac{1}{1}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$				
RHS:	$\frac{1}{1}$		$\frac{1}{3}$		$\frac{1}{5}$		$\frac{1}{7}$	
		$-\frac{1}{2}$		$-\frac{1}{4}$		$-\frac{1}{6}$		$-\frac{1}{8}$

[1] I learned this name for it from Svetoslav Savchev and Titu Andreescu, *Mathematical Miniatures*, Anneli Lax New Mathematical Library, Mathematical Association of America, 2003. (A wonderful book that all of you should buy immediately.) Be aware that the name "Catalan's identity" is also used for $F_n^2 - F_{n+k}F_{n-k} = (-1)^{n-k}F_k^2$ (where F_n is the nth Fibonacci number).

Now, we'd like to have $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{8}$ on the right; then we can match the whole first rows up. Add those terms in, and subtract them out again:

LHS:	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
	$-\frac{1}{1}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$				
RHS:	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
		$-\frac{1}{2}$		$-\frac{1}{4}$		$-\frac{1}{6}$		$-\frac{1}{8}$
		$-\frac{1}{2}$		$-\frac{1}{4}$		$-\frac{1}{6}$		$-\frac{1}{8}$

The rest of the pairing is easy to see: the first unaccounted term on the left, $-\frac{1}{1}$, goes with the first unaccounted pair on the right, the two copies of $-\frac{1}{2}$.

Putting this all together, we get the following argument:

$$\begin{aligned}
\sum_{1\le k\le 2n} \frac{(-1)^{k+1}}{k} &= \sum_{\substack{1\le k\le 2n\\ k \text{ odd}}} \frac{1}{k} - \sum_{\substack{1\le k\le 2n\\ k \text{ even}}} \frac{1}{k} \\
&= \sum_{\substack{1\le k\le 2n\\ k \text{ odd}}} \frac{1}{k} + \sum_{\substack{1\le k\le 2n\\ k \text{ even}}} \frac{1}{k} - \sum_{\substack{1\le k\le 2n\\ k \text{ even}}} \frac{1}{k} - \sum_{\substack{1\le k\le 2n\\ k \text{ even}}} \frac{1}{k} \\
&= \sum_{1\le k\le 2n} \frac{1}{k} - 2\sum_{\substack{1\le k\le 2n\\ k \text{ even}}} \frac{1}{k} \\
&= \sum_{1\le k\le 2n} \frac{1}{k} - 2\sum_{1\le k\le n} \frac{1}{2k} \\
&= \sum_{1\le k\le 2n} \frac{1}{k} - \sum_{1\le k\le n} \frac{1}{k} \\
&= \sum_{n+1\le k\le 2n} \frac{1}{k}
\end{aligned}$$

If the connection between the steps here and the previous fiddling-and-pairing-up is opaque, write each sum out explicitly; e.g., the first line is

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n} = \left(\frac{1}{1} + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$$

3 Matrix problems

A couple problems from our list:

1. Two parts:

   (a) Let A be an $m \times n$ matrix. Show that A^TA is symmetric; conclude that its eigenvalues are all real. Then show that they are all nonnegative.

The symmetry is easy: using the fact that $(AB)^T = B^T A^T$, with $A, B := A^T, A$, we get

$$(A^T A)^T = A^T (A^T)^T = A^T A\,.$$

As we know from linear algebra, a symmetric matrix's eigenvalues are all real.

Now, suppose x is an eigenvector of $A^T A$ with associated eigenvalue λ, that is,

$$A^T A x = \lambda x\,.$$

What can we do with that? Recalling $(AB)^T = B^T A^T$, we might think to multiply by x^T on both sides:

$$x^T A^T A x = x^T \lambda x\,;$$

in other words,

$$(Ax)^T (Ax) = \lambda x^T x\,;$$

in yet other words,

$$\|Ax\|^2 = \lambda \|x\|^2\,.$$

Now, since x is an eigenvector, it is nonzero; so its norm is nonzero too, and

$$\lambda = \frac{\|Ax\|^2}{\|x\|^2} \geq 0\,,$$

as claimed.

(b) Let M be an $n \times n$ symmetric matrix with nonnegative eigenvalues. Show that there exists a matrix A so that $M = A^T A$.

Since M is symmetric, it is orthogonally diagonalizable, that is, there exist an orthogonal matrix Q and a diagonal matrix D such that

$$M = Q^T D Q\,.$$

Moreover, the diagonal entries of D are the eigenvalues of M, which are nonnegative. Thus they have square roots; let C be the diagonal matrix whose entries are those square roots. Then $C^2 = D$ and $C^T = C$, so:

$$(CQ)^T (CQ) = Q^T C^T C Q = Q^T C^2 Q = Q^T D Q = M\,.$$

CQ is, then, the desired matrix.

2. What does adding I to a matrix do to its eigenvalues? Generalize.

Adding I to a matrix adds 1 to its eigenvalues. More precisely: x is an eigenvector of A with eigenvalue λ if and only if x is an eigenvector of $A + I$ with eigenvalue $\lambda + 1$. The proof is straightforward from the definitions. I'll leave the generalizations to you.