Math Club Notes: 2006 May 2

1 Estimating the sum of the reciprocals of the squares

One of Euler's famous results is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \,.$$

I don't know how this is proven. But we can get an estimate on this sum if we recall that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1 \,.$$

When the summand is expanded in partial fractions, we find that the sum telescopes: the partial sums are

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} \\ &= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} = 1 + \sum_{k=2}^{n} \frac{1}{k} - \sum_{k=2}^{n} \frac{1}{k} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

which tends to 1 as $n \to \infty$. (For other examples of telescoping sums, see our notes for 2005 May 22.)

So, for the original sum, we can compute

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \sum_{k=2}^{\infty} \frac{1}{k^2} &= 1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} \\ &< 1 + \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1 + 1 = 2 \,. \end{aligned}$$

Assuming you knew that $\sum 1/k(k+1) = 1$, how would you think of using it to study $\sum 1/k^2$? Well, we should always be on the lookout for related problems that we know how to solve; these problems are related in that they both involve summing the reciprocals of a quadratic function. Pursuing that thought, you might want to make up some other sums of this type to see what can be done with them. Perhaps $\sum 1/(k^2-1)$ and $\sum 1/(k^2+1)$, for example.

Another related problem whose solution we know: what's $\int_1^{\infty} x^{-2}\,dx$? Can we use that to say something about the sum?

2 Integrating secant

In Stewart we find the following derivation of the integral of secant:

$$\begin{aligned}\int \sec\theta\, d\theta &= \int \sec\theta \cdot \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\, d\theta \\ &= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\, d\theta\end{aligned}$$

Now recall that $\frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta$ and $\frac{d}{d\theta}\tan\theta = \sec^2\theta$. Thus, letting $u = \sec\theta+\tan\theta$,

$$\begin{aligned} &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\sec\theta+\tan\theta| + C\end{aligned}$$

This is correct, of course, but you might well wonder who thought of multiplying and dividing by $\sec\theta+\tan\theta$, and why.

Here's another, somewhat more natural, approach.[1] Earlier on the same page in Stewart where the derivation above occurs, we find the observation that, when faced with an integral of the form

$$\int \sin^n\theta\cos^m\theta\, d\theta$$

where m is odd, we can put one cos with the $d\theta$, convert the rest to sin using Pythagoras, and then make the substitution $u = \sin\theta$. Let's do that then.

$$\begin{aligned}\int \sec\theta\, d\theta &= \int \frac{1}{\cos\theta}\, d\theta && \text{(That's } (\cos\theta)^{-1}\text{, and } -1 \text{ is odd, so} \\ &= \int \frac{1}{\cos^2\theta}\cos\theta\, d\theta && \text{put one cos with the } d\theta, \\ &= \int \frac{1}{1-\sin^2\theta}\cos\theta\, d\theta && \text{convert the rest to sin,} \\ &= \int \frac{1}{1-u^2}\, du && \text{and substitute } u=\sin\theta.) \\ &= \frac{1}{2}\int\left(\frac{1}{1+u}+\frac{1}{1-u}\right) du \\ &= \frac{1}{2}(\ln|1+u| - \ln|1-u|) + C \\ &= \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C\end{aligned}$$

[1] According to Eli Maor, *Trigonometric Delights* (Princeton UP, 1998), p. 218–9, this was more or less how Isaac Barrow cracked this integral; his solution is notable for being the first recorded use of partial fractions.

So far, so good. But perhaps we want to simplify this. To get rid of the $\frac{1}{2}$, we could move it inside the logarithm, where it'll become a square root; so we'd like to see a perfect square inside the logarithm.

To do that, we'll undo the conversion from cos to sin: multiply and divide by $1 + \sin\theta$, the conjugate of the denominator.

$$\begin{aligned}
\frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C &= \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}\right| + C \\
&= \frac{1}{2}\ln\left|\frac{(1+\sin\theta)^2}{1-\sin^2\theta}\right| + C \\
&= \frac{1}{2}\ln\left|\frac{(1+\sin\theta)^2}{\cos^2\theta}\right| + C \\
&= \ln\left|\frac{1+\sin\theta}{\cos\theta}\right| + C \\
&= \ln|\sec\theta + \tan\theta| + C
\end{aligned}$$

Having reached this point, we might want to check the result, say, by differentiating this expression to verify that it is an antiderivative of secant. Once we've done that, we might notice that the calculation in this check can be reversed, yielding the quick but mysterious derivation we started with.

An alternative simplification can be had if we recall (from our notes for March 30) the identities

$$(\cos\tfrac{\theta}{2} \pm \sin\tfrac{\theta}{2})^2 = 1 \pm \sin\theta\,.$$

Thus

$$\begin{aligned}
\frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C &= \frac{1}{2}\ln\left|\frac{(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})^2}{(\cos\frac{\theta}{2}-\sin\frac{\theta}{2})^2}\right| + C \\
&= \ln\left|\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right| + C \\
&= \ln\left|\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right| + C
\end{aligned}$$

This is where you end up if your first move in the original problem is to make the Weierstrass substitution $t = \tan\frac{\theta}{2}$.