Math Club Notes: 2006 March 30

1 A triangle problem by Charles Dodgson

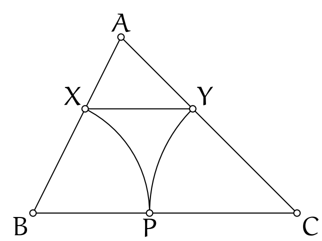
The problem:

> In a given Triangle to place a line parallel to the base, such that the portions of sides, intercepted between it and the base, shall be together equal to the base.[1]

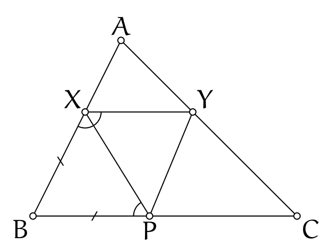
Another way to say it:

> Given $\triangle ABC$, find points X and Y on AB and AC respectively such that $XY \parallel BC$ and $BX + CY = BC$.

The most awkward condition is that $BX + CY = BC$. A natural way to dispose of it is to rotate BX and CY down onto the base BC. The condition will be satisfied iff X and Y come to rest on the same point on the base, say, P. So if we can find this point P then we'll be done.



How to find this point? A little more analysis is needed. Join PX and PY. Now: $\triangle BPX$ is isosceles, so $\angle BXP = \angle BPX$ (as angles opposite equal sides). Also, $XY \parallel BC$, so $\angle BPX = \angle PXY$ (as alternate interior angles). Therefore $\angle BXP = \angle PXY$, that is, XP is the bisector of $\angle BXY$.



Similarly YP is the bisector of $\angle CYX$. So XP and YP are the exterior bisectors of $\triangle AXY$; the interior bisector of $\angle A$ is concurrent with them, hence also passes

[1] Problem #2 from *Pillow Problems* by Charles L. Dodgson (aka Lewis Carroll), originally published 1895 by Macmillan, republished 1958 by Dover. Good ol' Dover.

through P. (Our notes for 2006 Jan 19 have a proof sketch of the concurrency of the interior bisectors; it is easily extended to this situation.)

So here's the construction: Bisect $\angle A$, and extend the bisector to intersect BC at P. Lay off $BX = BP$ and $CY = CP$ on the sides; join XY.

2 Parameterizing the unit circle

Last time I mentioned that the unit circle $\{(x, y) : x^2 + y^2 = 1\}$ can be parameterized by

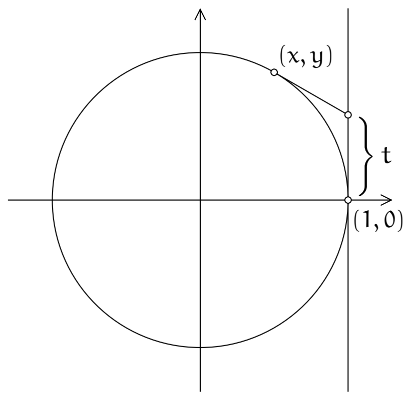
$$x = \frac{1 - t^2}{1 + t^2} \quad \text{and} \quad y = \frac{2t}{1 + t^2},$$

and asked if this t had any geometric interpretation (analogous to the familiar interpretation of θ as a central angle in the more familiar parameterization $(x, y) = (\cos\theta, \sin\theta)$).

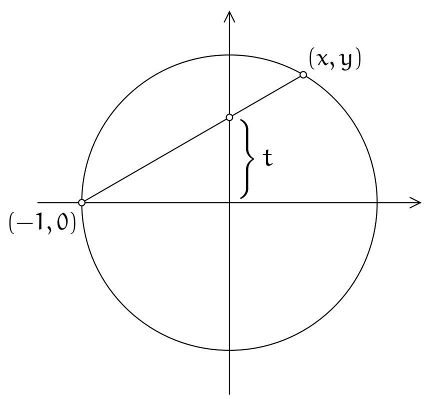
It turns out that $t = \tan\frac{\theta}{2}$, where θ is the familiar angle. The verification of the relevant identities

$$\cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} \quad \text{and} \quad \sin\theta = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$$

is left as an exercise. So are the proofs that the following two geometric interpretations of t are sound:



Draw a tangent to the unit circle at $(1, 0)$. Also draw a tangent at (x, y). Where these tangents intersect is t units up from the x-axis. (Of course, for points on the bottom half of the circle, the intersection is below the x-axis and t is negative.)



Join (x, y) to $(-1, 0)$. This line intersects the y-axis at a point t units up from the origin. (Again, for half the circle it's negative.)

3 Some trivial trigonometric inequalities

Let's prove that, for all θ,

$$-1 \leq \sin\theta \leq 1 .$$

(Why are we proving this familiar fact? Well, why not? Besides, we're going to do it in an unusual way.)

Here's one approach: for any $\theta \in \mathbb{R}$,

$$\begin{aligned}
& -1 \le \sin\theta \le 1 \\
\equiv & \qquad \{\text{algebra}\} \\
& \sin^2\theta \le 1 \\
\equiv & \qquad \{\text{algebra}\} \\
& 0 \le 1 - \sin^2\theta \\
\equiv & \qquad \{\text{Pythagorean identity}\} \\
& 0 \le \cos^2\theta \\
\equiv & \qquad \{\text{squares are nonnegative}\} \\
& \text{true}
\end{aligned}$$

("$\equiv$" here means "if and only if". The {blurbs in braces} are brief hints[2] about why the thing before and the thing after are logically equivalent.)

This is a silly example of a nevertheless useful and common technique for proving inequalities. Move everything to one side, so you're proving something of the form $0 \le A$; then re-write A as the square of something.

Above we first turned the two inequalities $-1 \le \sin\theta \le 1$ into the single inequality $\sin^2\theta \le 1$, to which we then applied this write-as-a-square technique. It turns out we can prove the two inequalities directly using the same technique. Note that

$$(\sin t - \cos t)^2 = \sin^2 t - 2\sin t\cos t + \cos^2 t = 1 - \sin 2t\ . \tag{1}$$

Thus, for any θ,

$$\begin{aligned}
& \sin\theta \le 1 \\
\equiv & \qquad \{\text{algebra}\} \\
& 0 \le 1 - \sin\theta \\
\equiv & \qquad \{\text{by (1), with } t = \tfrac{\theta}{2}\} \\
& 0 \le (\sin\tfrac{\theta}{2} - \cos\tfrac{\theta}{2})^2 \\
\equiv & \qquad \{\text{squares are nonnegative}\} \\
& \text{true}
\end{aligned}$$

By considering $(\sin t + \cos t)^2$ we get a similar proof that $-1 \le \sin\theta$.

Note, incidentally, the following:

$$\begin{aligned}
\cos^2 2t &= 1 - \sin^2 2t && \text{(Pythagorean identity)} \\
&= (1 - \sin 2t)(1 + \sin 2t) && \text{(difference of squares)}
\end{aligned}$$

[2] The word "algebra" might not seem like much of a hint. The point of such a hint is to tell the reader, not that algebra was used, but that *nothing but* algebra was used.

And this:

$$
\begin{aligned}
\cos^2 2t &= (\cos^2 t - \sin^2 t)^2 && \text{(double-angle identity)} \\
&= (\cos t - \sin t)^2 (\cos t + \sin t)^2 && \text{(difference of squares)}
\end{aligned}
$$

The identity (1) (and its analogue with $+$) tells us that these two ways of using difference of squares to factor $\cos^2 2t$ actually yield the same factorization.

4 Euclid's algorithm with linear algebra

(A favourite topic of mine.) Let's find $\gcd(700, 119)$, and express it as their linear combination.

$$
\begin{aligned}
\begin{bmatrix} 700 \\ 119 \end{bmatrix} &= \begin{bmatrix} 5 \cdot 119 + 105 \\ 119 \end{bmatrix} && (700 \div 119) \\
&= \begin{bmatrix} 5 \cdot 119 + 1 \cdot 105 \\ 1 \cdot 119 + 0 \cdot 105 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 119 \\ 105 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \cdot 105 + 14 \\ 105 \end{bmatrix} && (119 \div 105) \\
&= \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 105 \\ 14 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 14 \\ 7 \end{bmatrix} && (105 \div 14) \\
&= \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} && (14 \div 7)
\end{aligned}
$$

So $\gcd(700, 119) = 7$. Note now that all these 2×2 matrices are invertible, so we can move them over to the other side of the equality:

$$
\begin{aligned}
\begin{bmatrix} 7 \\ 0 \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 7 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 700 \\ 119 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 700 \\ 119 \end{bmatrix}
\end{aligned}
$$

At this point it is obvious that 7 is a linear combination of 700 and 119; linear combinations are what matrices do. Indeed,

$$\begin{aligned}
\begin{bmatrix} 7 \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 700 \\ 119 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -7 \\ -2 & 15 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 700 \\ 119 \end{bmatrix} \\
&= \begin{bmatrix} -7 & 8 \\ 15 & -17 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 700 \\ 119 \end{bmatrix} \\
&= \begin{bmatrix} 8 & -47 \\ -17 & 100 \end{bmatrix} \begin{bmatrix} 700 \\ 119 \end{bmatrix} \\
&= \begin{bmatrix} 8 \cdot 700 - 47 \cdot 119 \\ -17 \cdot 700 + 100 \cdot 119 \end{bmatrix}
\end{aligned}$$

and so $7 = 8 \cdot 700 - 47 \cdot 119$.

Isn't that much tidier than in the typical treatment?

Note also in the second half of the computation, where the coefficients of the linear combination are being found, how the first matrix on the various right-hand sides evolves from line to line. After considering that for a while, have a look at the following record of a computation:

700		47
119	5	8
105	1	7
14	7	1
7	2	0

The first and second columns are computed first, from top to bottom; then the third column is computed, from bottom to top. The greatest common divisor appears at the bottom left; the coefficients in the linear combination appear at the upper right.

Exercise: figure out the method by comparing this table to the previous matrix computation.

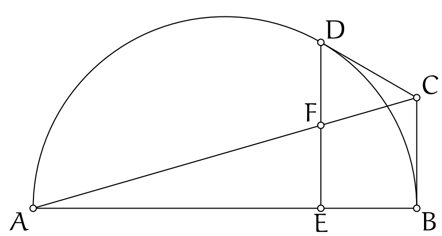
5 Problem roundup

Some old, some new. (Some borrowed, some blue.)

1. (Jan 19) Prove that $\vee$ (inclusive or) cannot be expressed using only $\neg$ and $\veebar$ (negation and exclusive or). [This problem was wrongly stated in the notes for Feb 9; the ors were reversed.]

2. (Feb 9) Prove that
$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=0}^{2n-1} \frac{(-1)^k}{k+1} .$$

3. (Mar 30) "Let AB be the diameter of a semicircle, and let the tangents to it at B and at any other point D on it meet in C. If now DE be drawn perpendicular to AB, and if AC, DE meet in F, then $DF = FE$."[3]



(Try to give both a trigonometric (q.v. section 2) and a geometric solution.)

4. (Mar 30) Given: n rectangles, each of area at least 1. Let R be the rectangle whose width is the average of the given rectangles' widths and whose height is the average of the given rectangles' heights. Prove that R has area at least 1.

5. (Mar 30) "Prove that 3 times the sum of 3 squares is also the sum of 4 squares."[4]

[3] This curious result is Proposition 2 from Archimedes' *Book of Lemmas*.
[4] Dodgson's problem #14, op. cit.. Try doing this one entirely in your head, as Dodgson suggests.