Math Club Notes: 2006 February 9

1 An inequality, solved

A solution to the problem posed last time: show by induction on n that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{3}{4}$$

for all integers $n \geq 1$.

For brevity, let S_n denote the sum in question; that is, for each integer $n \geq 1$, let

$$S_n = \sum_{k=1}^{n} \frac{1}{n+k} .$$

We will show that

$$S_n \leq \frac{3}{4} - \frac{1}{2n+2} .$$

In the base case $n = 1$, we have equality; for $S_1 = \frac{1}{2} = \frac{3}{4} - \frac{1}{4}$. Now suppose the inequality to be true for some integer $n \geq 1$; then

$$\begin{aligned}
S_{n+1} &= S_n + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\
&\leq \frac{3}{4} - \frac{1}{2n+2} + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} && \text{(by hypothesis)} \\
&= \frac{3}{4} + \frac{1}{2n+1} - \frac{1}{n+1} \\
&= \frac{3}{4} - \frac{n}{(2n+1)(n+1)} \\
&= \frac{3}{4} - \frac{n}{2n^2+3n+1} \\
&\leq \frac{3}{4} - \frac{n}{2n^2+3n+n} \\
&= \frac{3}{4} - \frac{n}{2n^2+4n} \\
&= \frac{3}{4} - \frac{1}{2n+4} \\
&= \frac{3}{4} - \frac{1}{2(n+1)+2}
\end{aligned}$$

which is the inequality for $n+1$. By induction, the inequality holds for all positive integers.

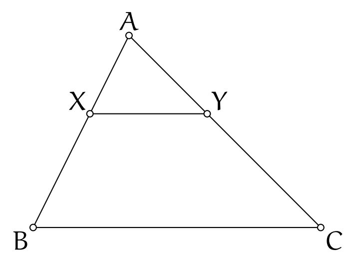
(That's the solution I came up with when I first saw this problem last year; it's different from the one we looked at tonight. Maybe we can look at this again next meeting.)

2 Problem roundup

Our current outstanding problems:

1. (Jan 19) Prove that $\veebar$ (exclusive or) cannot be expressed using only $\neg$ and $\vee$ (negation and inclusive or).

2. (Feb 9) A new geometry problem: "In a given Triangle to place a line parallel to the base, such that the portions of sides, intercepted between it and the base, shall be together equal to the base."[1] Another way to say it: given $\triangle ABC$, find points X and Y on AB and AC respectively such that $XY \parallel BC$ and $BX + CY = BC$.



3. (Feb 9) Recall that the unit circle $x^2 + y^2 = 1$ has this parametric representation:

$$x = \cos\theta \qquad\qquad y = \sin\theta$$

Here θ can be interpreted as the angle formed between the x-axis and the ray from the origin through the point (x, y). Here's another parameterization of the unit circle:

$$x = \frac{1 - t^2}{1 + t^2} \qquad\qquad y = \frac{2t}{1 + t^2}$$

The question: what is the geometric meaning of t? (Eileen and Martha recognized these expressions as having something to do with some kind of integration. Track that down and you'll be well on your way to solving the problem.)

4. (Feb 9) My notes on the inequality in section 1 also mention, without proof, that

$$S_n = \sum_{k=0}^{2n-1} \frac{(-1)^k}{k+1} .$$

Take a stab at proving this, if you like. (Note that it then follows $\ln 2 < \frac{3}{4}$; see our notes for 2005 June 13.)

[1] Problem #2 from *Pillow Problems* by Charles L. Dodgson (aka Lewis Carroll), originally published 1895 by Macmillan, republished 1958 by Dover. Good ol' Dover.