Math Club Notes: 2006 January 19

1 A trigonometric/hyperbolic identity

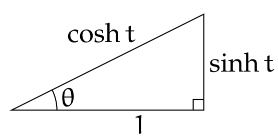
A problem I brought to a previous meeting:[1] show that

$$\arctan \sinh t = \arcsin \tanh t$$

for all real t. We saw three proofs.

1.1 Martha's proof

Consider a right triangle with legs of length 1 and $\sinh t$. By Pythagoras's theorem, the hypotenuse has length $\sqrt{1+\sinh^2 t}$, which by a hyperbolic identity is $\cosh t$.



Let θ be the angle adjacent the leg of length 1. Then

$$\sin\theta = \frac{\sinh t}{\cosh t} = \tanh t\,,$$

while

$$\tan\theta = \frac{\sinh t}{1} = \sinh t\,.$$

Thus $\arcsin \tanh t = \theta = \arctan \sinh t$.

(Does this proof work when $\sinh t < 0$? What we should perhaps do is consider, instead of a right triangle and its lengths, the line $x = 1$ and two parameterizations for it, one in terms of t and one in terms of θ.)

1.2 Eileen's proof

Let f and g be the two functions in question:

$$f(t) = \arcsin \tanh t \qquad \text{and} \qquad g(t) = \arctan \sinh t\,.$$

Now, recalling (or deriving afresh) that

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\sinh x = \cosh x$$
$$\operatorname{sech}^2 x = 1 - \tanh^2 x \qquad \cosh^2 x = 1 + \sinh^2 x$$

[1] James Stewart, *Calculus*, 5th ed., page 508.

we can compute the derivatives of f and g by the chain rule:

$$f'(t) = \frac{1}{\sqrt{1-\tanh^2 t}} \cdot \operatorname{sech}^2 t = \frac{\operatorname{sech}^2 t}{\sqrt{\operatorname{sech}^2 t}} = \operatorname{sech} t$$
$$g'(t) = \frac{1}{1+\sinh^2 t} \cdot \cosh t = \frac{\cosh t}{\cosh^2 t} = \operatorname{sech} t$$

So f and g have the same derivative; thus they differ by a constant. By evaluating $f(0) = 0 = g(0)$, we see that that constant is zero. So they are the same function.

1.3 Steven's proof

We wish to show that the functions $\arcsin \tanh x$ and $\arctan \sinh x$ are equal, that is,

$$\sin^{-1} \circ \tanh = \tan^{-1} \circ \sinh \ .$$

Apply $\tan$ from the left and $\sinh^{-1}$ from the right to obtain the equivalent equality

$$\tan \circ \sin^{-1} \circ \tanh \circ \sinh^{-1} = I \,,$$

where I denotes the identity function. Parenthesized as

$$(\tan \circ \sin^{-1}) \circ (\tanh \circ \sinh^{-1}) = I \,,$$

this new equality states that the functions $\tan \circ \sin^{-1}$ and $\tanh \circ \sinh^{-1}$ are inverses.

Well,

$$\tan \arcsin x = \frac{\sin \arcsin x}{\cos \arcsin x} = \frac{\sin \arcsin x}{\sqrt{1-\sin^2 \arcsin x}} = \frac{x}{\sqrt{1-x^2}}$$
$$\tanh \operatorname{arcsinh} x = \frac{\sinh \operatorname{arcsinh} x}{\cosh \operatorname{arcsinh} x} = \frac{\sinh \operatorname{arcsinh} x}{\sqrt{1+\sinh^2 \operatorname{arcsinh} x}} = \frac{x}{\sqrt{1+x^2}}$$

A little algebra verifies that, indeed, these functions are inverses.

2 Proving concurrency

Consider a triangle $\triangle ABC$, and the following three (sketches of) proofs.

1. The perpendicular bisector of AB is the locus of points equidistant from the vertices A and B. Similarly, the perpendicular bisector of BC is the locus of points equidistant from the vertices B and C. So their intersection X is the same distance from A as from B, and the same distance from B as from C. Therefore X is the same distance from A as from C, whence it lies on the perpendicular bisector of AC. So all three perpendicular bisectors pass through this point X, that is, they concur.

2. Inside $\triangle ABC$, the bisector of $\angle A$ is the locus of points equidistant from the sides AB and AC. Similarly, the bisector of $\angle B$ is the locus of points equidistant from the sides AB and BC. So their intersection X is the same distance from AB as from AC, and the same distance from AB as from BC. Therefore X is the same distance from AC as from BC, whence it lies on the bisector of $\angle C$. So all three bisectors pass through point X, that is, they concur.

3. Inside $\triangle ABC$, the median from A (that is, the line joining A to the midpoint of the opposite side) is the locus of points P such that $\triangle APB$ and $\triangle APC$ have the same area. Similarly, the median from B is the locus of points P such that $\triangle APB$ and $\triangle BPC$ have the same area. So their intersection X forms triangles with the vertices such that $\triangle AXB$ and $\triangle AXC$ have the same area, and $\triangle AXB$ and $\triangle BXC$ have the same area. Therefore $\triangle AXC$ and $\triangle BXC$ have the same area, whence X lies on the median from C. So all three medians pass through point X, that is, they concur.

(Note that the latter two specify "inside $\triangle ABC$". This is necessary. For there are points equidistant from AB and AC, outside $\triangle ABC$, which do not lie on the bisector of $\angle A$. Those points form external bisectors at that vertex; the full locus consists of two perpendicular lines meeting at A. Exercise: what is the full locus of points P such that $\triangle AXB$ and $\triangle AXC$ have the same area?)

These three proofs have the same structure; the concurrency of three lines is proven from, essentially, the transitivity of equality.

The question: can a similar proof be given of the concurrence of the altitudes?

3 The ors

Let $\vee$ denote inclusive or; let $\veebar$ denote exclusive or. Thus we have the following truth table (where 0 represents false and 1 represents true):

A	B	$A \vee B$	$A \veebar B$
1	1	1	0
1	0	1	1
0	1	1	1
0	0	0	0

Also let $\neg$ denote "not", and $\wedge$ denote "and".

In De Morgan's law, $\vee$ is expressed using just $\neg$ and $\wedge$:

$$A \vee B \equiv \neg(\neg A \wedge \neg B) \ .$$

$\vee$ can also be expressed using just $\wedge$ and $\veebar$:

$$A \vee B \equiv A \veebar B \veebar (A \wedge B) \ .$$

(No more parentheses are needed on the right because, as it turns out, $\veebar$ is associative.)

Can $\vee$ be expressed using just $\veebar$ and $\neg$? Turns out no, though proving it may take some trouble.

4 An inequality

Prove, by induction on n, that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{3}{4}$$

for all $n \geq 1$.

(The problem is somewhat badly phrased. You are to prove the above inequality by induction; that is not meant to imply that this inequality itself must be your inductive hypothesis. You may prove a related statement by induction instead.)