Math Club Notes: 2006 January 12

1 Sum of every third binomial coefficient, again

An alternative method[1] for one of our past problems, namely finding a closed form for $\sum_k \binom{n}{3k}$. (See our notes for 2005 Oct 27.)

Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Note that 1, ω, and ω^2 are the three complex solutions of the equation $z^3 = 1$. Now, by the binomial theorem,

$$\begin{aligned}
(1+1)^n &= \sum_k \binom{n}{k} 1^k = \sum_k \binom{n}{3k} + \sum_k \binom{n}{3k+1} + \sum_k \binom{n}{3k+2} \\
(1+\omega)^n &= \sum_k \binom{n}{k} \omega^k = \sum_k \binom{n}{3k} + \sum_k \binom{n}{3k+1}\omega + \sum_k \binom{n}{3k+2}\omega^2 \\
(1+\omega^2)^n &= \sum_k \binom{n}{k} \omega^{2k} = \sum_k \binom{n}{3k} + \sum_k \binom{n}{3k+1}\omega^2 + \sum_k \binom{n}{3k+2}\omega
\end{aligned}$$

Adding these three equations together, we obtain

$$2^n + (1+\omega)^n + (1+\omega^2)^n = 3\sum_k \binom{n}{3k},$$

since $1 + \omega + \omega^2 = 0$. Now,

$$\begin{aligned}
(1+\omega)^n &= (\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i)^n \\
&= (\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3})^n \\
&= \cos\tfrac{n\pi}{3} + i\sin\tfrac{n\pi}{3} && \text{(De Moivre's theorem)}
\end{aligned}$$

and similarly

$$(1+\omega^2)^n = \cos\tfrac{n\pi}{3} - i\sin\tfrac{n\pi}{3}.$$

Putting it all together,

$$\sum_k \binom{n}{3k} = \tfrac{1}{3}(2^n + 2\cos\tfrac{n\pi}{3}),$$

which is equivalent to the first closed form we found before.

This method generalizes somewhat more easily than the previous method; you might want to try working out $\sum_k \binom{n}{4k}$, for example, or even (if feeling adventurous) $\sum_k \binom{n}{mk}$. (Or see the footnote.)

[1] This solution is a (very) special case of exercise 38 from section 1.2.6 of D. E. Knuth, *The Art of Computer Programming: Fundamental Algorithms*, 3rd ed. (Reading: Addison-Wesley, 1997), pp. 71, 487.

2 Continued square root

Again, an alternative method[2] for one of our past problems, namely evaluating

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}\,.$$

(See our notes for 2005 July 11.) As in our previous solution, let's define the sequence $(a_n)_{n\geq 0}$ by the recurrence

$$\begin{aligned} a_0 &= 0 \\ a_{n+1} &= \sqrt{2+a_n} \end{aligned}$$

Then it is natural to say that

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}} = \lim_{n\to\infty} a_n\,,$$

if this limit exists. (In fact this is just about the only reasonable way to define the infinite expression in question.)

Suppose that for some n and θ, with $-\pi < \theta < \pi$, we have

$$a_n = 2\cos\theta\,.$$

Then

$$\begin{aligned} a_{n+1} &= \sqrt{2+2\cos\theta} \\ &= 2\sqrt{\frac{1+\cos\theta}{2}} \\ &= 2\cos\frac{\theta}{2}\,. \end{aligned}$$

(We need $-\pi < \theta < \pi$ to be sure that $\cos(\theta/2)$ is nonnegative, hence equal to this square root.) Since $a_0 = 2\cos\frac{\pi}{2}$, a simple induction shows that

$$a_n = 2\cos\frac{\pi}{2^{n+1}}\,,$$

whence ($2\cos x$ being a continuous function of x)

$$\lim_{n\to\infty} a_n = 2\cos 0 = 2\,.$$

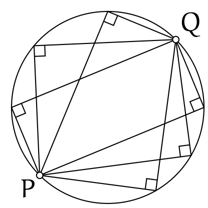
Slick, eh? Alas, it does not generalize well; e.g., if we had 3s instead of 2s, this method wouldn't even get off the ground.

(How did anybody ever come up with this? Probably they were looking at the values $\cos(\pi/2^n)$ for some other reason — e.g., in connection with inscribing regular polygons of an ever-doubling number of sides in a circle. Indeed, Gelfand and Saul use this result to approximate π.)

[2] I found this solution in I. M. Gelfand and M. Saul, *Trigonometry* (Boston: Birkhäuser 2001), p. 163–166.

3 An algebraic identity concerning circles

Given points P and Q. For which points X do we have $\angle PXQ = 90°$?



We know the answer to this question from geometry: the locus of X is a circle with PQ as diameter. What does this fact look like algebraically?

Consider the following identity:

$$(x-p)(x-q) = \left(x - \frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2 .$$

To verify this, you could factor the right-hand side as a difference of squares; or, if you prefer, you could expand the left-hand side and complete the square.

Now consider the following identity (where $\langle\ ,\ \rangle$ represents an inner product, such as the dot product, and $\|\ \|$ represents the associated norm):

$$\langle \vec{x}-\vec{p}, \vec{x}-\vec{q}\rangle = \|\vec{x} - \tfrac{1}{2}(\vec{p}+\vec{q})\|^2 - \|\tfrac{1}{2}(\vec{p}-\vec{q})\|^2 .$$

The algebra to verify this identity is *exactly the same* as the algebra to verify the previous identity, just with the inner product instead of the usual product. (Note how much nicer it is to expand $\|\vec{a}\|^2$ as $\langle \vec{a}, \vec{a}\rangle$ than as $\|\vec{a}\|\cdot\|\vec{a}\|$; the former has far better algebraic properties.)

From this identity it immediately follows that the statements

$$\langle \vec{x}-\vec{p}, \vec{x}-\vec{q}\rangle = 0 \qquad \text{and} \qquad \|\vec{x} - \tfrac{1}{2}(\vec{p}+\vec{q})\| = \|\tfrac{1}{2}(\vec{p}-\vec{q})\|$$

are equivalent. The former states that the lines joining point $\vec{x}$ to points $\vec{p}$ and $\vec{q}$ are orthogonal. The latter states that $\vec{x}$ lies on a circle centred at $\frac{1}{2}(\vec{p}+\vec{q})$ and of radius $\|\frac{1}{2}(\vec{p}-\vec{q})\|$, which is exactly the circle with the line segment joining $\vec{p}$ and $\vec{q}$ as a diameter; this circle is, in fact, the intersection of the paraboloid $z = \langle \vec{x}-\vec{p}, \vec{x}-\vec{q}\rangle$ with the xy-plane, in just the same manner as the points $x = p$ and $x = q$ are the intersection of the parabola $y = (x-p)(x-q)$ with the x-axis.

4 Happy New Year!

$2006 = 17 \times 17 + 1717$.