Math Club Notes: 2005 December 8

We looked at two of this year's Putnam problems today.

1 A polynomial

The problem (Putnam 2005, B1):

> Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a.

This is actually a pretty easy problem, if you have any experience with the floor function, which it is convenient to define this way: for any real x and integer n,

$$n = \lfloor x \rfloor \equiv n \le x < n + 1 \; . \tag{1}$$

We are dealing with $\lfloor a \rfloor$ and $\lfloor 2a \rfloor$. How are these related? Is the latter just twice the former? Let's see:

$$\begin{aligned}
& 2\lfloor a \rfloor = \lfloor 2a \rfloor \\
\equiv \quad & \{(1), \text{with } n, x := 2\lfloor a \rfloor, 2a\} \\
& 2\lfloor a \rfloor \le 2a < 2\lfloor a \rfloor + 1 \\
\equiv \quad & \{\text{algebra}\} \\
& \lfloor a \rfloor \le a < \lfloor a \rfloor + \tfrac{1}{2} \\
\equiv \quad & \{\text{algebra}\} \\
& 0 \le a - \lfloor a \rfloor < \tfrac{1}{2}
\end{aligned}$$

So, they're not the same in general, but we now know exactly when they are the same: when the fractional part of a — that is, the amount by which a exceeds the next integer down — is between 0 and $\frac{1}{2}$. (Note that, when $a < 0$, its "fractional part" might not be what you expect; for example, the fractional part of -3.2 is not 0.2, but $-3.2 - \lfloor -3.2 \rfloor = -3.2 - (-4) = 0.8$. "Rounding down" means "towards $-\infty$", not "towards 0".)

What about the other possibility? Let's see:

$$\begin{aligned}
& \tfrac{1}{2} \le a - \lfloor a \rfloor < 1 \\
\equiv \quad & \{\text{algebra}\} \\
& 1 \le 2a - 2\lfloor a \rfloor < 2 \\
\equiv \quad & \{\text{algebra}\} \\
& 2\lfloor a \rfloor + 1 \le 2a < 2\lfloor a \rfloor + 2 \\
\equiv \quad & \{(1)\} \\
& 2\lfloor a \rfloor + 1 = \lfloor 2a \rfloor
\end{aligned}$$

In summary, $\lfloor 2a \rfloor$ is either $2\lfloor a \rfloor$ or $2\lfloor a \rfloor + 1$, depending on the fractional part of a.

So what we want is that

$$P(n, 2n) = 0 \qquad \text{and} \qquad P(n, 2n+1) = 0$$

for all integers n. We notice that

$$(x, y) = (n, 2n) \implies 2x - y = 0$$

and

$$(x, y) = (n, 2n+1) \implies 2x - y + 1 = 0$$

So here's our polynomial:

$$P(x, y) = (2x - y)(2x - y + 1) .$$

Whatever a is, one of these factors will come out to be zero when we evaluate $P(\lfloor a \rfloor, \lfloor 2a \rfloor)$.

2 An integral

The problem (2005 Putnam A5) is to evaluate the integral

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx .$$

(You might want to refresh your memory of $\int_0^\pi \ln \sin x \, dx$, which we looked at in the summer — see the notes for July 4.)

The first bit of my solution was just getting the integral into a form I was more comfortable with. First, integrate by parts with

$$u = \ln(x+1) \qquad\qquad v = \arctan x$$

$$du = \frac{dx}{x+1} \qquad\qquad dv = \frac{dx}{x^2+1}$$

to determine that

$$\begin{aligned}\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx &= \ln(x+1)\arctan x\Big|_0^1 - \int_0^1 \frac{\arctan x}{x+1}\,dx \\ &= \frac{\pi}{4}\ln 2 - \int_0^1 \frac{\arctan x}{x+1}\,dx\end{aligned}$$

Doesn't look much better, but let's take a stab at this new integral. First try to get rid of the arctan (because inverse trig functions scare me): let $\theta = \arctan x$. Then $x = \tan\theta$, so $dx = \sec^2\theta\, d\theta$, and

$$\int_0^1 \frac{\arctan x}{x+1}\,dx = \int_0^{\pi/4} \frac{\theta \sec^2\theta}{1+\tan\theta}\,d\theta$$

No more inverse trig, but a bunch more trig. Secant also scares me; I prefer cosine. So multiply and divide by $\cos^2\theta$:

$$= \int_0^{\pi/4} \frac{\theta}{\cos^2\theta + \sin\theta\cos\theta}\,d\theta$$

"$\sin\theta\cos\theta$" looks familiar. Let's multiply and divide by 2 and use the double-angle identities.

$$\begin{aligned}
&= \int_0^{\pi/4} \frac{2\theta}{2\cos^2\theta + 2\sin\theta\cos\theta}\,d\theta \\
&= \int_0^{\pi/4} \frac{2\theta}{1 + \cos 2\theta + \sin 2\theta}\,d\theta
\end{aligned}$$

That's a lot of 2θ. Let $a = 2\theta$.

$$= \frac{1}{2}\int_0^{\pi/2} \frac{a}{1 + \cos a + \sin a}\,da$$

Call this expression I.

Notice the nice symmetry between cos and sin in this integral. Anything that exchanges cos and sin will leave the denominator of this integrand unchanged…

Substitute $b = \frac{\pi}{2} - a$ (so that also $a = \frac{\pi}{2} - b$).

$$\begin{aligned}
I &= \frac{1}{2}\int_{\pi/2}^{0} \frac{\frac{\pi}{2} - b}{1 + \cos(\frac{\pi}{2} - b) + \sin(\frac{\pi}{2} - b)}(-\,db) \\
&= -\frac{1}{2}\int_{\pi/2}^{0} \frac{\frac{\pi}{2} - b}{1 + \cos(\frac{\pi}{2} - b) + \sin(\frac{\pi}{2} - b)}\,db \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{\frac{\pi}{2} - b}{1 + \cos(\frac{\pi}{2} - b) + \sin(\frac{\pi}{2} - b)}\,db \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{\frac{\pi}{2} - b}{1 + \sin b + \cos b}\,db \qquad \text{(complementary angles)} \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{\frac{\pi}{2} - b}{1 + \cos b + \sin b}\,db \\
&= \frac{\pi}{4}\int_0^{\pi/2} \frac{1}{1 + \cos b + \sin b}\,db - \frac{1}{2}\int_0^{\pi/2} \frac{b}{1 + \cos b + \sin b}\,db \\
&= \frac{\pi}{4}\int_0^{\pi/2} \frac{1}{1 + \cos b + \sin b}\,db - I
\end{aligned}$$

Thus

$$I = \frac{\pi}{8}\int_0^{\pi/2} \frac{1}{1 + \cos b + \sin b}\,db\,.$$

The b in the numerator (well, it was an a back then) has disappeared. Integrals involving a mixture of functions of different types — e.g., trigonometric functions and polynomials — are often more difficult than integrals involving just one type. We just got rid of the polynomial part of our integrand; what's left is just trig. Major progress.

Now reverse the steps that brought us here:

$$\begin{aligned}
I &= \frac{\pi}{8}\int_0^{\pi/2} \frac{1}{1+\cos b+\sin b}\,db \\
&= \frac{\pi}{8}\int_0^{\pi/4} \frac{2}{1+\cos 2t+\sin 2t}\,dt && (t=\tfrac{1}{2}b) \\
&= \frac{\pi}{8}\int_0^{\pi/4} \frac{2}{2\cos^2 t+2\sin t\cos t}\,dt \\
&= \frac{\pi}{8}\int_0^{\pi/4} \frac{1}{\cos^2 t+\sin t\cos t}\,dt \\
&= \frac{\pi}{8}\int_0^{\pi/4} \frac{\sec^2 t}{1+\tan t}\,dt && (\text{multiply/divide by } \sec^2 t) \\
&= \frac{\pi}{8}\int_0^{1} \frac{1}{1+u}\,du && (u=\tan t) \\
&= \frac{\pi}{8}\Big[\ln|1+u|\Big]_0^1 \\
&= \frac{\pi}{8}\ln 2
\end{aligned}$$

(Note how at the last, the arctan is gone and life has become easy.) Substituting back into where we started, we get

$$\begin{aligned}
\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx &= \frac{\pi}{4}\ln 2 - \frac{\pi}{8}\ln 2 \\
&= \frac{\pi}{8}\ln 2\,.
\end{aligned}$$

If you know your trig identities better than I do, you can use the idea here much earlier in the computation; see the solution by Bhargava, Kedlaya, and Ng at http://www.unl.edu/amc/a-activities/a7-problems/putnam/ to see how that looks. (A key element in their solution is

$$\cos\theta+\sin\theta = \sqrt{2}\cos(\tfrac{\pi}{4}-\theta)\,,$$

which arises from the more familiar identity

$$\cos\alpha\cos\beta+\sin\alpha\sin\beta = \cos(\alpha-\beta)$$

by taking $\alpha = \frac{\pi}{4}$.)