Math Club Notes: 2005 December 1

We looked at some old Putnam problems.

1 A rational/irrational series

The problem:

> For each $k \in \mathbb{N}$, let
> $$Q_k = \frac{1}{(k+2)!} + \frac{2}{(k+3)!} + \frac{3}{(k+4)!} + \cdots .$$
> Prove that Q_k is rational if and only if $k = 0$.

So we're dealing with
$$Q_k = \sum_{n \geq 1} \frac{n}{(n+k+1)!} .$$
Note, incidentally, that although each term in the sum is rational, so each partial sum is rational, their limit could be rational or irrational.

I don't really want to talk about this sum; let's talk instead about the function
$$Q_k(x) = \sum_{n \geq 1} \frac{nx^{n-1}}{(n+k+1)!} .$$
Note that $Q_k = Q_k(1)$.

(Introducing powers of x like this is a reasonably common maneuver when dealing with sums. As we will see, it makes some otherwise challenging manipulations very easy. For more on strategies for evaluating sums, see *Concrete Mathematics*, by Graham, Knuth, and Patashnik.)

(We should perhaps take a moment to prove that this power series converges. I'll just say that factorial beats everything, so it converges.)

Now for some sum manipulation. That nx^{n-1} is familiar.
$$Q_k(x) = \frac{d}{dx} \sum_{n \geq 1} \frac{x^n}{(n+k+1)!} .$$
(This is a usual way to destroy an n in a sum; it's also why we put in x^{n-1} instead of x^n. But if you put in x^n instead, you could just factor an x out of the sum and then do this.) (How would you destroy a $1/n$?)

Having gotten rid of the n, we now see that the terms are a power of x over a factorial. That reminds us of
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{n \geq 0} \frac{x^n}{n!} .$$

So let's try to get to something more like this sum for e^x.

$$\begin{aligned}
Q_k(x) &= \frac{d}{dx}\sum_{n\geq 1}\frac{x^n}{(n+k+1)!} \\
&= \frac{d}{dx}\sum_{n\geq k+2}\frac{x^{n-k-1}}{n!} && \text{(replace } n \text{ with } n-k-1\text{)} \\
&= \frac{d}{dx}\frac{1}{x^{k+1}}\sum_{n\geq k+2}\frac{x^n}{n!} \\
&= \frac{d}{dx}\frac{1}{x^{k+1}}\left(e^x - \sum_{n=0}^{k+1}\frac{x^n}{n!}\right)
\end{aligned}$$

Excellent. Note that the remaining sum is finite — so, unlike when we started, if each term is rational then that sum is too. Can we go back to Q_k now, that is, take $x = 1$? Not quite — we have to differentiate first. So let's do that:

$$Q_k(x) = \frac{1}{x^{k+1}}\left(e^x - \sum_{n=1}^{k+1}\frac{x^{n-1}}{(n-1)!}\right) - \frac{k+1}{x^{k+2}}\left(e^x - \sum_{n=0}^{k+1}\frac{x^n}{n!}\right) .$$

And then take $x = 1$:

$$\begin{aligned}
Q_k &= Q_k(1) \\
&= e - \sum_{n=1}^{k+1}\frac{1}{(n-1)!} - (k+1)\left(e - \sum_{n=0}^{k+1}\frac{1}{n!}\right) \\
&= -ke - \sum_{n=1}^{k+1}\frac{1}{(n-1)!} + (k+1)\sum_{n=0}^{k+1}\frac{1}{n!}
\end{aligned}$$

and, though this expression can be further simplified, for present purposes we might as well stop here.

Everything except the ke is rational; so Q_k is rational if and only if ke is. Since e is irrational, ke is rational if and only if $k = 0$. And we're done.

2 An enumeration of the positive rationals

The problem:

> Consider the recurrence
>
> $$\begin{aligned} a_0 &= 1 \\ a_{2n+1} &= a_n \\ a_{2n+2} &= a_n + a_{n+1} \end{aligned}$$
>
> Show that every positive rational occurs in the set
>
> $$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \dots \right\} .$$

(Eileen noticed that we could use this result to show that $\mathbb{Q}$ is countable.)

The recurrence tells us about the sequence (a_n), but we wish to prove something about the sequence (a_{n-1}/a_n). Give that a name: let

$$b_n = \frac{a_{n-1}}{a_n} .$$

The recurrence relates values of a near $2n$ to values of a near n. Can we get similar relations for b? Yes:

$$\begin{aligned} b_{2n+1} &= \frac{a_{2n}}{a_{2n+1}} \\ &= \frac{a_{n-1} + a_n}{a_n} \\ &= \frac{a_{n-1}}{a_n} + 1 \\ &= b_n + 1 , \end{aligned}$$

and

$$\begin{aligned} b_{2n+2} &= \frac{a_{2n+1}}{a_{2n+2}} \\ &= \frac{a_n}{a_n + a_{n+1}} \\ &= \frac{a_n/a_{n+1}}{a_n/a_{n+1} + 1} \\ &= \frac{b_{n+1}}{b_{n+1} + 1} . \end{aligned}$$

What do these relations tell us? If $b_n = p/q$, then

$$b_{2n+1} = b_n + 1 = \frac{p}{q} + 1 = \frac{p+q}{q}$$

and

$$b_{2n} = \frac{b_n}{b_n+1} = \frac{p/q}{p/q+1} = \frac{p}{p+q} \; .$$

So if we add the numerator to the denominator, we stay in the set; also if we add the denominator to the numerator.

Note that p/q is in some sense a "smaller" fraction than $(p+q)/q$ and $p/(p+q)$. Make that precise, and we have an inductive proof that every positive rational occurs in the set, as follows.

In what follows, p and q always denote positive integers.

Suppose that every positive rational number p/q with $\max(p,q) \le M$ is in the set, and consider a positive rational number p/q with $\max(p,q) = M+1$.

If $p > q$, then $p = \max(p,q) = M+1$. Also, $(p-q)/q$ is positive and rational; since q is positive, $p-q < p = M+1$, and by hypothesis $q < p = M+1$, so $\max(p-q,q) \le M$. By the inductive hypothesis, $(p-q)/q$ is in the set, say, $b_n = (p-q)/q$. Then $p/q = b_{2n+1}$ is also in the set.

If, on the other hand, $p < q$, then by a similar argument $p/(q-p)$ is also in the set, say, $b_n = p/(q-p)$. Then $p/q = b_{2n}$ is also in the set.

Finally, if $p = q$, then $p/q = 1 = b_1$ is in the set.

By induction, every positive rational is in the set.

An alternative way to present this proof is by infinite descent: argue that if p/q is not in the set, then either (for $p < q$) also $p/(q-p)$ is not in the set, or (for $p > q$) also $(p-q)/q$ is not in the set. Repeat. Since $\max(p,q)$ decreases when we subtract the lesser from the greater, and remains positive, this process cannot continue forever; it must terminate, with $p = q$. But then we have shown that $p/q = 1$ is not in the set, which is false.

Expressed this way, it's perhaps a bit easier that the reduction step is just Euclid's algorithm. This sequence is thus closely related to the Farey tree, to continued fractions, and all kinds of nifty stuff.

3 An eventually periodic sequence of integers

The problem:

> Let $(p_n)_{n\ge 1}$ be a bounded sequence of integers satisfying
>
> $$p_n = \frac{p_{n-1} + p_{n-2} + p_{n-3}p_{n-4}}{p_{n-1}p_{n-2} + p_{n-3} + p_{n-4}} \; .$$
>
> Show that the sequence is eventually periodic.

There's not much to this. Since the sequence is bounded, it contains only finitely many values, say, k of them. A run of four consecutive values, then, can take at most k^4 different forms; the point is, that's finite. So some such run occurs twice. Since each p_n depends only on the previous four terms, at that point the sequence will begin repeating itself.

The tricky bit (such as it is) is just that the details of the expression given are irrelevant.

4 Finding a minimum

The problem:

> Find the minimum value of
>
> $$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$
>
> for $x > 0$.

We could, of course, differentiate and set the derivative to zero, etc.; but we are not eager to differentiate this hideous thing. Can we simplify it first?

Turns out yes. The trick is to recall the identity

$$(a+1/a)^2 = a^2+2+1/a^2 \,.$$

("Recall"? When have we seen this? Well, I saw it in Math 115, when calculating arclengths. In order for the arclength integral to come out nicely, the authors of such problems have to make a certain expression come out as a perfect square. This is one of the perfect squares they often use.)

Thus we can write our expression as

$$\frac{(x+1/x)^6-(x^3+1/x^3)^2}{(x+1/x)^3+(x^3+1/x^3)} \,.$$

Let $a = (x+1/x)^3$ and $b = x^3+1/x^3$ to see that the numerator is just a^2-b^2 and the denominator is $a+b$. Thus their quotient is $a-b$, that is,

$$(x+1/x)^3-(x^3+1/x^3) \,.$$

Multiply out the first term and cancel a couple things, and the expression becomes

$$3x+3/x \,.$$

Now it's certainly feasible to differentiate, etc., as usual. Alternatively, since we're only interested in $x > 0$, we can apply the inequality of the arithmetic and geometric means: we have

$$3x+3/x = \tfrac{1}{2}(6x+6/x) \geq \sqrt{6x\cdot 6/x} = 6 \,,$$

with equality exactly when $3x = 3/x$, that is, $x = 1$. So the minimum value of this expression (for $x > 0$) is 6.

5 The eighth root of a continued fraction

The problem:

> Express
> $$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cdots}}}$$
> in the form $(a + b\sqrt{c})/d$, where a, b, c, and d are integers.

Let's assume for the moment that the infinite expression under the radical converges. Then we can do our usual trick: call the eighth root x; then raising both sides to the eighth power yields

$$x^8 = 2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cdots}} ,$$

whence (since the denominator of the topmost fraction is the same as the whole right-hand side)

$$x^8 = 2207 - \frac{1}{x^8} . \tag{1}$$

(We did something like this once before, to evaluate $\sqrt{2 + \sqrt{2 + \cdots}}$; see our notes for July 11.)

Now, the obvious thing to do at this point is to multiply through by x^8 and rearrange to get

$$x^{16} - 2207x^8 + 1 = 0 ,$$

which is a quadratic in x^8. Applying the quadratic formula, we get

$$x^8 = \tfrac{1}{2}(2207 \pm \sqrt{2207^2 - 4}) .$$

That gives x^8 in the required form, but we want x in the required form, and the best we can do in this approach is to say

$$x = \sqrt[8]{\tfrac{1}{2}(2207 \pm \sqrt{2207^2 - 4})} ,$$

which is not what we want.

What else can we do? From (1) we have

$$x^8 + \frac{1}{x^8} = 2207 .$$

That left-hand side looks a little like the expressions from the problem in the previous section. So we can do this:

$$(x^4 + 1/x^4)^2 = x^8 + 2 + 1/x^8 = 2209 .$$

Wouldn't it be nice if 2209 were a perfect square? Turns out it's 47^2. So

$$x^4 + 1/x^4 = 47 .$$

Now do the same thing again, to get

$$x^2 + 1/x^2 = 7 ,$$

and then again, to get

$$x + 1/x = 3 .$$

And *now* we'll multiply by x and solve the resulting quadratic, to get

$$x = \tfrac{1}{2}(3 \pm \sqrt{5}) .$$

Much progress.

But is it $+$ or $-$? Both of the values $3 \pm \sqrt{5}$ are positive, so the fact that x is supposed to be an eighth root doesn't help. Turns out we have to understand the continued fraction a little better.

About the only good way to define the expression $2207 - 1/(2207 - \cdots)$ is as the limit of the sequence defined by the recurrence

$$\begin{aligned} a_0 &= 2207 \\ a_{n+1} &= 2207 - 1/a_n \end{aligned}$$

if that limit exists. (Expand the first few terms in this sequence to see why this is a good definition.)

A quick way to show that this sequence converges: define $f(x) = 2207 - 1/x$. Since $f'(x) = 1/x^2 > 0$ for all x, this function is strictly increasing everywhere, that is, $a < b \equiv f(a) < f(b)$. The point, of course, is that $a_{n+1} = f(a_n)$. So, once we calculate that $a_1 = 2207 - 1/2207 < 2207 = a_0$, we reason that

$$a_1 < a_0 \equiv f(a_1) < f(a_0) \equiv a_2 < a_1 .$$

Similarly, $a_3 < a_2$, and $a_4 < a_3$, and so on. (Insert inductive proof here.)

So the sequence (a_n) is strictly decreasing; if it's also bounded below, it converges. This is not too hard: we have $a_n > 2206$ for all n. This is immediate for a_0; for a_{n+1} we have, by induction,

$$a_{n+1} = 2207 - 1/a_n > 2207 - 1/2206 > 2207 - 1 = 2206 .$$

What have we learned? The sequence (a_n) converges, so we can assign a value to the continued fraction, namely the limit of that sequence. In fact, that limit is at least 2206 (since each value in the sequence is greater than 2206).

This last observation lets us determine whether the eighth root is $\frac{1}{2}(3 + \sqrt{5})$ or $\frac{1}{2}(3 - \sqrt{5})$. The latter is less than 1, so its eighth power is also less than 1, which is certainly less than 2206. So it's got to be $\frac{1}{2}(3 + \sqrt{5})$, which completes the solution.

6 A weird recurrence

The problem:

> $(x_n)_{n\geq 0}$ is a sequence of nonzero real numbers satisfying
> $$x_n^2 - x_{n-1}x_{n+1} = 1$$
> for all $n \geq 1$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

6.1 Solving the problem

We proceed by establishing that what we want to prove is equivalent to something which is obviously true. Note that, in what follows, we need to have $\equiv$ (that is, logical equivalence), or at least $\Leftarrow$ (that is, "follows from") at every step. Having $\Rightarrow$ is useless; we don't care what follows from what we're trying to prove, but only what it follows from.

For convenience, let $f(n) = (x_{n+1} + x_{n-1})/x_n$. (We'll see why this is convenient in a minute.)

Now to the problem.

$$\begin{array}{ll}
& \exists a\colon \forall n\colon x_{n+1} = ax_n - x_{n-1} \\
\equiv & \quad \{\text{solve for } a\text{; note that } x_n \neq 0\} \\
& \exists a\colon \forall n\colon (x_{n+1} + x_{n-1})/x_n = a \\
\equiv & \quad \{\text{definition of } f\} \\
& \exists a\colon \forall n\colon f(n) = a \\
\equiv & \quad \{\text{two fancy ways of saying } f \text{ is constant}\} \qquad (2) \\
& \forall m, n\colon f(m) = f(n) \\
\equiv & \quad \{\Rightarrow \text{ by taking } m = n+1\text{; } \Leftarrow \text{ by induction}\} \\
& \forall n\colon f(n) = f(n+1) \\
\equiv & \quad \{\text{definition of } f\} \\
& \forall n\colon (x_{n+1} + x_{n-1})/x_n = (x_{n+2} + x_n)/x_{n+1} \\
\equiv & \quad \{\text{algebra; note that } x_n \neq 0 \text{ and } x_{n+1} \neq 0\} \\
& \forall n\colon x_{n+1}^2 + x_{n-1}x_{n+1} = x_n^2 + x_n x_{n+2} \\
\equiv & \quad \{\text{algebra}\} \qquad (3) \\
& \forall n\colon x_{n+1}^2 - x_n x_{n+2} = x_n^2 - x_{n-1}x_{n+1} \\
\equiv & \quad \{\text{hypothesis, for } n \text{ and } n+1\} \\
& \forall n\colon 1 = 1 \\
\equiv & \quad \{= \text{ is reflexive}\} \\
& \text{true}
\end{array}$$

This solution is pretty straightforward, meaning that each step is the most natural thing to do. (Step (2), using two ways of saying a function is constant, we have seen before; see Gowers's derivation of the Cauchy-Schwarz inequality in our notes for May 22. Step (3) is motivated by wanting to make use of the hypothesis — the $x_{n-1}x_{n+1}$ belongs with the x_n^2, etc.)

Note, incidentally, that the 1 is not important; all that matters is that it's a constant.

6.2 How did they know?

The whole approach in the previous section relies on suspecting that there is a recurrence of the form $x_{n+1} = ax_n - x_{n-1}$. Once we have that conjecture, proving it is not too hard. But how would we know there was such a recurrence in the first place? How did the authors of the problem know?

Here's a way one could find out there's such a recurrence without too many lucky guesses. Imagine we are studying sequences $(x_n)_{n\geq 0}$ satisfying

$$x_n^2 - x_{n-1}x_{n+1} = 1$$

for all $n \geq 1$, and we simply wish to know whether anything interesting can be said about them. (One such sequence is $(0, 1, 3, 8, 21, \dots)$, which consists of every other Fibonacci number. This sequence has one zero in it, but it turns out not to matter.)

First, we notice that what we have is the value of a determinant:

$$\begin{vmatrix} x_n & x_{n+1} \\ x_{n-1} & x_n \end{vmatrix} = 1\,. \tag{4}$$

And it's awfully structured, this determinant; it's not just four random values. Each row consists of two consecutive elements of the sequence, and the top row is just shifted one over from where the bottom row is.

What should the next row up be? Just continue the pattern:

$$\begin{vmatrix} x_{n+1} & x_{n+2} \\ x_n & x_{n+1} \\ x_{n-1} & x_n \end{vmatrix}$$

The bottom two rows here form a determinant of the type we know something about — the determinant in (4). And so do the top two rows — they're just (4) for $n+1$ instead of n.

Of course, a 3×2 determinant doesn't make sense — determinants have to be square. So we should add a third column.

$$\begin{vmatrix} \alpha & x_{n+1} & x_{n+2} \\ \beta & x_n & x_{n+1} \\ \gamma & x_{n-1} & x_n \end{vmatrix}$$

It's not clear yet what values we want in that column.

What we know about this 3×3 determinant is the values of two of the 2×2 subdeterminants. How do we relate 3×3 determinants to their 2×2 subdeterminants? By the Laplace expansion. Taking the expansion along the first column, and using the fact that some of the subdeterminants are 1, we have

$$\begin{vmatrix} \alpha & x_{n+1} & x_{n+2} \\ \beta & x_n & x_{n+1} \\ \gamma & x_{n-1} & x_n \end{vmatrix} = \alpha - \beta \begin{vmatrix} x_{n+1} & x_{n+2} \\ x_{n-1} & x_n \end{vmatrix} + \gamma \, .$$

We don't know much about that other 2×2 determinant, so let's just get rid of it: take $\beta = 0$.

$$\begin{vmatrix} \alpha & x_{n+1} & x_{n+2} \\ 0 & x_n & x_{n+1} \\ \gamma & x_{n-1} & x_n \end{vmatrix} = \alpha + \gamma \, .$$

What about α and γ? We want to set them so that... what? What *do* we want this 3×3 determinant to be? Well, it could be anything, but we know that all kinds of special things happen when a determinant is 0, so let's make it 0. That is, we'll take $\alpha = -\gamma$; say, $\alpha = 1$ and $\gamma = -1$.

So now we're looking at the fact that, for a recurrence satisfying (4), we have

$$\begin{vmatrix} 1 & x_{n+1} & x_{n+2} \\ 0 & x_n & x_{n+1} \\ -1 & x_{n-1} & x_n \end{vmatrix} = 0 \, .$$

One thing this means is that the three columns are linearly dependent. The first two, however, are independent (if we assume $x_n \neq 0$); so the third column lies in the span of the first two:

$$\begin{bmatrix} x_{n+2} \\ x_{n+1} \\ x_n \end{bmatrix} \in \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} x_{n+1} \\ x_n \\ x_{n-1} \end{bmatrix} \right\} .$$

Note that the vector on the left and the vector on the right are of the same type — both consist of three consecutive elements of our sequence. Taking the elements of that sequence three at a time, we get a sequence of vectors in $\mathbb{R}^3$; what we now know is that each of those vectors lies in the span of this fixed vector $(1, 0, -1)^T$ and the previous vector in the sequence.

An easy induction then shows that

$$\begin{bmatrix} x_{n+1} \\ x_n \\ x_{n-1} \end{bmatrix} \in \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix} \right\} .$$

What is this span? A plane in $\mathbb{R}^3$. So it has a normal vector, that is, a vector to which everything in the plane is orthogonal, including these vectors consisting of elements of the sequence. Let's compute a normal vector, say, the cross product of our two basis vectors for the plane:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_0 - x_2 \\ x_1 \end{bmatrix} .$$

The fact that this is orthogonal to each vector in our sequence is just the statement that

$$\begin{bmatrix} x_1 \\ -x_0 - x_2 \\ x_1 \end{bmatrix} \cdot \begin{bmatrix} x_{n+1} \\ x_n \\ x_{n-1} \end{bmatrix} = 0$$

for all $n \geq 1$. Writing that out in detail, we get

$$x_1 x_{n+1} - (x_0 + x_2) x_n + x_1 x_{n-1} = 0 ,$$

and solving for x_{n+1} yields

$$x_{n+1} = \left(\frac{x_0 + x_2}{x_1} \right) x_n - x_{n-1} ,$$

a recurrence such as the original problem described.