Math Club Notes: 2005 October 13

1 Functions with an unusual property

The aforementioned U of Waterloo contest has the following problem:

> Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}$ with the property that
> $$\forall x, y \in \mathbb{R}^+\colon f(x+y) = f(x^2+y^2)\ .$$

We had previously noticed that constant functions have this property; today we looked at a sketch of a proof that they are the only such functions.

In other words, we are proving the theorem

$$(\forall x, y \in \mathbb{R}^+\colon f(x+y) = f(x^2+y^2)) \iff f \text{ is constant}\ .$$

(Here and in what follows, we take f to have type $\mathbb{R}^+ \to \mathbb{R}$; we won't keep mentioning this fact.) It is useful in this problem to have a clean definition of "constant function"; the nicest one I know is

$$f \text{ is constant} \iff (\forall a, b \in \mathbb{R}^+\colon f(a) = f(b))\ .$$

So what we're proving is that (for functions of the type under consideration)

$$\forall x, y \in \mathbb{R}^+\colon f(x+y) = f(x^2+y^2) \tag{1}$$

is equivalent to

$$\forall a, b \in \mathbb{R}^+\colon f(a) = f(b) \tag{2}$$

The $\Leftarrow$ direction is easy. Suppose (2), and let $x, y \in \mathbb{R}^+$; we must show that $f(x+y) = f(x^2+y^2)$. Just take $a, b := x+y, x^2+y^2$ in (2).

The $\Rightarrow$ direction is not so easy. Suppose (1), and let $a, b \in \mathbb{R}^+$; we must show that $f(a) = f(b)$. Now, the only thing we know about f is (1). Fortunately, (1) has approximately the same form as what we're trying to prove: we wish to show that the value of f is the same at some two points, and that's what (1) says. So, we dream a bit: wouldn't it be nice if we could find x and y such that $x+y = a$ and $x^2+y^2 = b$? Then we would have a proof, which would look like this:

> Suppose (1), and let $a, b \in \mathbb{R}^+$. Let
> $$x = ?$$
> $$\text{and } y = ?$$
> Then
> $$x + y = a$$
> $$\text{and } x^2 + y^2 = b\ ,$$
> so $f(a) = f(x+y) = f(x^2+y^2) = f(b)$. Therefore f is constant.

So now we just need to fill in those question marks.

In other words, we have reduced the problem to this one:

Given $a, b \in \mathbb{R}^+$, find $x, y \in \mathbb{R}^+$ such that

$$x + y = a$$
$$x^2 + y^2 = b$$

That is, we wish to solve this system for x and y. (It will turn out that we can't always do this, so our plan will have to be revised.)

Deploying our usual techniques: From the first equation we have $y = a - x$; use this to eliminate y from the second equation, obtaining

$$x^2 + (a - x)^2 = b\ .$$

Expand and rearrange to obtain

$$2x^2 - 2ax + (a^2 - b) = 0\ ,$$

which is quadratic in x. Apply the quadratic formula to obtain

$$x = \tfrac{1}{2}a + \tfrac{1}{2}\sqrt{2b - a^2}\ ,$$

and then use $y = a - x$ to obtain

$$y = \tfrac{1}{2}a - \tfrac{1}{2}\sqrt{2b - a^2}\ .$$

(The quadratic formula actually has $\pm$, so we could have chosen $-$ in the expression for x; then we'd have $+$ in the expression for y. Since x and y enter symmetrically into the system we're solving, it doesn't matter which way we do this.)

It is straightforward to verify that, with these values of x and y, we do indeed have $x + y = a$ and $x^2 + y^2 = b$. (For the latter, it is handy to apply the identity

$$(u + v)^2 + (u - v)^2 = 2u^2 + 2v^2$$

with $u, v := \frac{1}{2}a, \frac{1}{2}\sqrt{2b - a^2}$.)

But there is a problem: we require that x and y be positive real numbers. In order for them to be real, we'll have to be able to take the square root $\sqrt{2b - a^2}$, that is, we require that

$$2b - a^2 \geq 0\ .$$

If that holds, then x will be positive (since a is positive, and square roots are at least zero); but for y to be positive we will further require that

$$\tfrac{1}{2}a > \tfrac{1}{2}\sqrt{2b - a^2}\ .$$

Applying a little algebra to simplify these inequalities, then combining them, we find that our values for x and y will both be positive reals if and only if

$$\tfrac{1}{2}a^2 \leq b < a^2\ . \qquad (3)$$

Alas, this inequality will not be satisfied for all $a, b \in \mathbb{R}^+$.

Let's take stock. We were trying to obtain x and y solving a certain system; given such values, we could show that $f(a) = f(b)$. But in order to obtain such values, we required that (3) be satisfied. So what we have shown is this:

If $a, b \in \mathbb{R}^+$ satisfy the inequality $\frac{1}{2}a^2 \le b < a^2$, then $f(a) = f(b)$.

This is not quite what we had hoped to show, but, as it turns out, it's enough.

Here's why this is enough: Consider a to be fixed. If b satisfies (3), then we can obtain $f(a) = f(b)$. That is, all through the interval $[\frac{1}{2}a^2, a^2)$, the value of f is the same as it is at a. So f is constant in this interval.

Now increase a a little bit. We get a new interval $[\frac{1}{2}a^2, a^2)$, on which f is again constant. Since we increased a only a little bit, this new interval will overlap the old interval; so f is constant on their union. By fiddling with a in this manner, we can get a whack of overlapping intervals covering the whole of $\mathbb{R}^+$, and conclude that f is constant on $\mathbb{R}^+$.

To do this formally, we want to construct a (doubly-infinite) sequence of positive reals

$$\ldots, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, \ldots$$

such that the associated intervals

$$\ldots, [\tfrac{1}{2}a_{-2}^2, a_{-2}^2), [\tfrac{1}{2}a_{-1}^2, a_{-1}^2), [\tfrac{1}{2}a_0^2, a_0^2), [\tfrac{1}{2}a_1^2, a_1^2), [\tfrac{1}{2}a_2^2, a_2^2), \ldots$$

cover all of $\mathbb{R}^+$, and overlap "enough" — say, each overlaps with the previous interval and the succeeding one.

Getting them to overlap is not so hard. To get the left endpoint of the interval for a_{n+1} to lie inside the interval for a_n, we want

$$\tfrac{1}{2}a_n^2 \le \tfrac{1}{2}a_{n+1}^2 < a_n^2 \,,$$

which (everything being positive) readily simplifies to

$$a_n \le a_{n+1} < \sqrt{2}a_n \,.$$

So it would work if a_{n+1} was, say, the geometric mean of a_n and $\sqrt{2}a_n$:

$$a_{n+1} = \sqrt[4]{2}a_n \,.$$

So here's one way to define our sequence:

$$a_n = 2^{n/4} \,.$$

Then each associated interval $[\frac{1}{2}a_n^2, a_n^2)$ overlaps with the interval for $n-1$ and with the interval for $n+1$, and together these intervals cover all of $\mathbb{R}^+$: as $n \to \infty$, the right endpoints grow without bound, and as $n \to -\infty$, the left endpoints converge to 0. (Writing out the details left as an exercise.)

2 Solutions to 2005 IMO problems

A fellow over at U of Connecticut posted some solutions to the 2005 IMO problems: http://www.math.uconn.edu/~rafi/Thought/Thought.html.