Math Club Notes: 2005 August 22

We looked at a couple problems from the 2005 International Mathematical Olympiad today. (The IMO is a competition for high school students.)

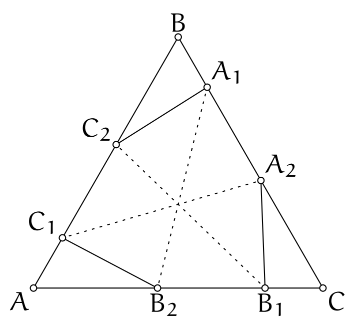
1 Hexagon in triangle

The problem (paraphrased):

> $\triangle ABC$ is equilateral. Points A_1, A_2 lie on BC, points B_1, B_2 lie on CA, and points C_1, C_2 lie on AB, forming a convex equilateral hexagon $A_1A_2B_1B_2C_1C_2$. Prove that A_1B_2, B_1C_2, and C_1A_2 are concurrent.

Note that an equilateral hexagon (i.e., one whose sides are all equal) need not be equiangular. Note also that the convexity of the hexagon tells us about the order of its vertices around the triangle.

Here's a picture:



We are given that the triangle is equilateral, that is,

$$AB = BC = CA ,$$

and that the hexagon is equilateral, that is,

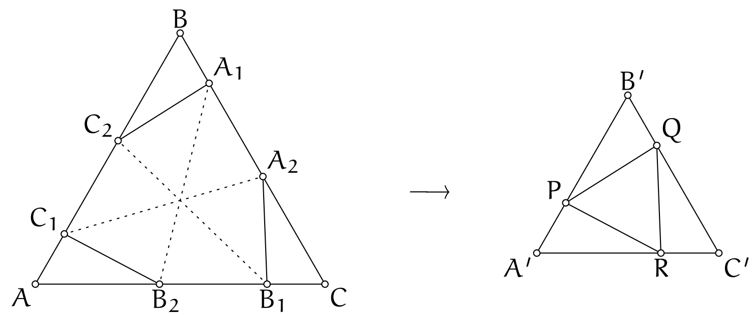
$$A_1A_2 = A_2B_1 = B_1B_2 = B_2C_1 = C_1C_2 = C_2A_1 .$$

We are to show that the dashed lines are concurrent, as in the figure. (From convexity and the givens, we also have some betweenness relations, which we'll just encode in the figure. A complete solution would explain, for example, how we know that B, A_1, A_2, C occur in that order on their line.)

We solved this problem by a fairly complicated argument; we suspect that it can be simplified. (The argument I'll give here is actually not quite what we came up with in the meeting, but it's quite similar.)

1.1 Corner triangles are congruent

One of our early conjectures about the figure was that the corner triangles ($\triangle AB_2C_1$ and its analogues) are congruent. Proving this required a clever trick: removing the hexagon.



In these figures, the removal of the hexagon carries A, B, C to A', B', C' respectively. It collapses A_1 and A_2 into the single point Q; B_1 and B_2 into R; and C_1 and C_2 into P. Intuitively, we just push the corner triangles together. (We'll make that more rigorous in a minute.)

Assuming this hexagon-removing maneuver works the way we intuitively expect, we reason that

$$\begin{aligned}
& \angle A'PR \\
= \quad & \{\text{sum of interior angles in } \triangle A'PR\} \\
& 180^\circ - \angle PRA' - \angle RA'P \\
= \quad & \{\triangle A'B'C' \text{ is equilateral}\} \\
& 180^\circ - \angle PRA' - 60^\circ \\
= \quad & \{\triangle PQR \text{ is equilateral}\} \\
& 180^\circ - \angle PRA' - \angle QRP \\
= \quad & \{\angle C'RQ, \angle QRP, \angle PRA' \text{ together form straight angle } \angle A'RC'\} \\
& \angle C'RQ
\end{aligned}$$

Similarly, $\angle PRA' = \angle RQC'$, and since $PR = RQ$ (as sides of the equilateral hexagon), by ASA we have $\triangle A'PR = \triangle C'RQ$ ($= \triangle B'QP$, similarly).

To make the hexagon-removing maneuver more formal, we need to construct the right-hand figure somehow, then prove that it has all the properties we want. Our intuitive conception of the figure is that it results from pushing the corner triangles together; to move a triangle from one place to another, we typically construct a new triangle in such a way that it is congruent to the old triangle. (For example, we have a construction by SSS, that is, a method to construct a triangle with three given sides, provided that those sides satisfy the triangle inequality.)

Perhaps the most natural way to express "pushing the corner triangles together" as a construction is this: We know that $C_2A_1 = A_2B_1 = B_2C_1$ (as sides of the equilateral hexagon); construct an equilateral triangle $\triangle PQR$ with sides of that same length. Then construct $\triangle A'PR$ on PR so that $\triangle A'PR = \triangle AB_2C_1$, and similarly for the other two corners.

Now, what properties do we need the new figure to have?

We want to carry conclusions deduced in the right-hand figure back to the left-hand figure; so we'll need the new corner triangles ($\triangle A'PR$, etc.) to be congruent to the old corner triangles ($\triangle AC_1B_2$, etc.). This is so by construction.

We used the fact that $\triangle A'B'C'$ is equilateral; this follows from its being equiangular, which follows from the fact that its angles are the images of the original angles of $\triangle ABC$, which are all equal to each other. So for this it's enough that the new and old corner triangles are congruent (so the 60° angles are preserved).

We used the fact that $\triangle PQR$ is equilateral; this follows from the fact that its sides are the images of some sides of the hexagon, and the hexagon is equilateral. So again, it is enough that the new and old corner triangles are congruent.

Slightly more subtly, we used the fact that P, Q, and R lie on $A'B'$, $B'C'$, and $C'A'$, respectively. This is, in fact, crucial — it's the very assumption that it is possible to assemble the corner triangles into the new figure. And it turns out to be tricky to prove; I don't see how to show that our construction makes A', P, and B' collinear.

A typical technique in this kind of situation: invert the construction. That is, construct the figure so that the problematic property is true by construction, and then prove the other properties instead.

So, a new construction for the right-hand figure: since $AB = BC = CA$ (as sides of the equilateral triangle) and $C_1C_2 = A_1A_2 = B_1B_2$ (as sides of the equilateral hexagon), by subtracting the latter from the former we deduce that

$$AC_1 + C_2B = BA_1 + A_2C = CB_1 + B_2A\ .$$

Construct an equilateral triangle $\triangle A'B'C'$ with sides of that same length. Then lay off $A'P = AC_1$, whence $PB' = C_2B$. This way, $\triangle A'B'C'$ is equilateral by construction, P lies on $A'B'$ by construction, and $A'P$ and PB' have the right lengths by construction (and similarly for the other two sides).

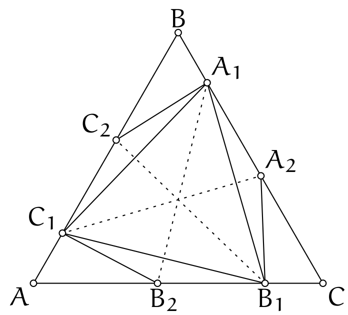
We then need to show that the new corner triangles are congruent to the old corner triangles. We have $A'P = AC_1$ by construction, and $A'R = AB_2$ by construction, and $\angle A' = 60^\circ = \angle A$ since $\triangle A'B'C'$ is equilateral by construction and $\triangle ABC$ is equilateral by hypothesis. Thus, by SAS, $\triangle A'RP = \triangle AB_2C_1$, as desired. (And similarly for the other two corners.)

Finally, that $\triangle PQR$ is equilateral follows, as before, from the congruence of the new and old corner triangles.

That completes the construction of the new figure, which justifies our proof that in that figure the corner triangles are congruent to each other, which implies they are congruent in the original figure, as desired.

1.2 Concurrence

We noticed that we can show that the hexagon's "main diagonals" concur by showing that they are medians or bisectors or altitudes of some triangle. (We take as known the theorems that medians, etc., concur.) Which triangle? Medians, bisectors, or altitudes, each line will have to pass through one vertex of the triangle; so take one endpoint from each line, making, say, $\triangle A_1B_1C_1$.



We will only show that A_1B_2 is a bisector of $\triangle A_1B_1C_1$; the proofs for the other two lines are similar. Finding the following partial argument is fairly routine:

$$
\begin{aligned}
& A_1B_2 \text{ is a bisector of } \triangle A_1B_1C_1 \\
\equiv \quad & \{\text{definition of "bisector"}\} \\
& \angle B_1A_1B_2 = \angle C_1A_1B_2 \\
\Leftarrow \quad & \{\text{corresponding angles}\} \\
& \triangle B_1A_1B_2 = \triangle C_1A_1B_2 \\
\equiv \quad & \{\text{SSS}\} \\
& B_1A_1 = C_1A_1 \text{ and } A_1B_2 = A_1B_2 \text{ and } B_2B_1 = B_2C_1
\end{aligned}
$$

(The relevant heuristic: to show that two angles are congruent, show that they are corresponding angles in some two congruent triangles.)

The second conjunct, $A_1B_2 = A_1B_2$, is trivial (congruence is reflexive). The third, $B_2B_1 = B_2C_1$, is given (these are two sides of our equilateral hexagon). So all we need to show is the first: $B_1A_1 = C_1A_1$. We find the following partial argument (by the same heuristic as before, for segments instead of angles):

$$
\begin{aligned}
& B_1A_1 = C_1A_1 \\
\Leftarrow \quad & \{\text{corresponding sides}\} \\
& \triangle A_1A_2B_1 = \triangle C_1C_2A_1 \\
\equiv \quad & \{\text{SAS}\} \\
& A_1A_2 = C_1C_2 \text{ and } \angle A_1A_2B_1 = \angle C_1C_2A_1 \text{ and } A_2B_1 = C_2A_1
\end{aligned}
$$

The first and third conjuncts are given (sides of the equilateral hexagon), so we need only show the second: $\angle A_1A_2B_1 = \angle C_1C_2A_1$. And that's easy: these are supplements of $\angle CA_2B_1$ and $\angle BC_2A_1$, which are equal by the congruence of the corner triangles.

2 Contest combinatorics

The second IMO problem we looked at (paraphrased again):

> A competition was held in which each contestant attempts to solve 6 problems. No contestant solved all 6. Every pair of problems was solved by more than $\frac{2}{5}$ of the contestants. Show that there are at least two contestants who solved exactly 5 of the problems.

We didn't solve this; we were able to prove only that there is at least *one* contestant who solved exactly 5 problems. The methods we used for this weaker result are still of some interest.

First, some notation. Sums will be represented as

$$(\Sigma x\colon f(x))$$

The outer parentheses show the scope of the dummy variable x, which will always have some implied range of admissible values; we sum over all such values.

Note in particular that $(\Sigma x\colon 1)$ denotes the number of admissible values of x. We will also use the notation

$$[P] = \begin{cases} 0 & \text{if } P \text{ is false} \\ 1 & \text{if } P \text{ is true} \end{cases}$$

which has the convenient property that $(\Sigma x\colon [P(x)])$ denotes the number of values of x such that $P(x)$ is true.

In analogy to the notation for sums, we will write $(\forall x\colon P(x))$ and $(\exists x\colon P(x))$ for universal and existential quantification. We will also use

$$(\text{Min } x\colon f(x))$$

to denote the minimum value of $f(x)$ as x ranges over its values. We will only need two intuitively obvious properties of minima:

$$(\text{Min } x\colon f(x)) \le y \;\equiv\; (\exists x\colon f(x) \le y) \tag{1}$$

and

$$(\text{Min } x\colon f(x)) \le (\text{Avg } x\colon f(x)) \tag{2}$$

where averages are defined by

$$(\text{Avg } x\colon f(x)) = (\Sigma x\colon f(x))/(\Sigma x\colon 1) \tag{3}$$

(That is, the average is the sum, divided by the number of values.)

Now to the problem.

As stated, the problem asks us to prove

$$\mathsf{P} \text{ and } \mathsf{Q} \Rightarrow \mathsf{R}\,,$$

where (for the weaker result that we were able to show)

$$\begin{aligned}
\mathsf{P} &\equiv \text{“no contestant solved all 6 problems”}\\
\mathsf{Q} &\equiv \text{“every pair of problems was solved by} > \tfrac{2}{5} \text{ of the contestants”}\\
\mathsf{R} &\equiv \text{“there is at least one contestant who solved exactly 5 problems”}
\end{aligned}$$

The obvious way to do this is to suppose P and Q, then deduce R. We will instead prove the theorem in the equivalent form

$$\mathsf{P} \text{ and } \neg\mathsf{R} \Rightarrow \neg\mathsf{Q}\,,$$

that is, we will suppose P and $\neg\mathsf{R}$ and deduce $\neg\mathsf{Q}$.

Since there are only 6 problems altogether, P and $\neg\mathsf{R}$ are together equivalent to the statement that no contestant solved more than 4 problems. Thus no contestant solved more than $\binom{4}{2} = 6$ pairs of problems.

Now, let c have type “contestant”, so that sums, quantifications, etc. over c range over all the contestants. Let N denote the number of contestants; that is,

$$\mathsf{N} = (\Sigma \mathsf{c}\colon 1) \tag{4}$$

Let t have type “pair of problems”, so that sums, etc., over t range over all distinct pairs of problems, of which there are $\binom{6}{2} = 15$. We have supposed that

$$\begin{aligned}
&\text{“no contestant solved more than 6 pairs of problems”}\\
\equiv\quad &\{\text{formalization}\}\\
&\neg(\exists \mathsf{c}\colon (\Sigma \mathsf{t}\colon [\mathsf{c} \text{ solved } \mathsf{t}]) > 6)\\
\equiv\quad &\{\text{logic}\}\\
&(\forall \mathsf{c}\colon (\Sigma \mathsf{t}\colon [\mathsf{c} \text{ solved } \mathsf{t}]) \le 6)
\end{aligned} \tag{5}$$

(Note that “c solved t” means that c solved both of the problems of which the pair t consists.) What we wish to show, namely $\neg\mathsf{Q}$, is

$$\begin{aligned}
&\neg \text{ “every pair of problems was solved by} > \tfrac{2}{5} \text{ of the contestants”}\\
\equiv\quad &\{\text{formalization}\}\\
&\neg(\forall \mathsf{t}\colon (\Sigma \mathsf{c}\colon [\mathsf{c} \text{ solved } \mathsf{t}]) > \tfrac{2}{5}\mathsf{N})\\
\equiv\quad &\{\text{logic}\}\\
&(\exists \mathsf{t}\colon (\Sigma \mathsf{c}\colon [\mathsf{c} \text{ solved } \mathsf{t}]) \le \tfrac{2}{5}\mathsf{N})\\
\equiv\quad &\{(1)\}\\
&(\mathrm{Min}\ \mathsf{t}\colon (\Sigma \mathsf{c}\colon [\mathsf{c} \text{ solved } \mathsf{t}])) \le \tfrac{2}{5}\mathsf{N}
\end{aligned} \tag{6}$$

To prove (6) by using (5), we must relate the sum over c in (6) to the sum over t in (5); we achieve that as follows:

$$\begin{aligned}
& (\text{Min } t\colon (\Sigma c\colon [c \text{ solved } t])) \\
\leq \quad & \{(2)\} \\
& (\text{Avg } t\colon (\Sigma c\colon [c \text{ solved } t])) \\
= \quad & \{\text{definition of average}\} \\
& (\Sigma t\colon (\Sigma c\colon [c \text{ solved } t]))/(\Sigma t\colon 1) \\
= \quad & \{\text{there are 15 pairs of problems}\} \\
& (\Sigma t\colon (\Sigma c\colon [c \text{ solved } t]))/15 \\
= \quad & \{\text{exchange sums}\} \\
& (\Sigma c\colon (\Sigma t\colon [c \text{ solved } t]))/15
\end{aligned}$$

Thus we have

$$(\text{Min } t\colon (\Sigma c\colon [c \text{ solved } t])) \leq (\Sigma c\colon (\Sigma t\colon [c \text{ solved } t]))/15\,, \tag{7}$$

and applying (5) then yields

$$(\text{Min } t\colon (\Sigma c\colon [c \text{ solved } t])) \leq (\Sigma c\colon 6)/15 = \tfrac{2}{5}N\,,$$

as desired.

Attempting to apply the same method to the stronger result, that there are at least *two* contestants who solved exactly 5 problems, we would suppose (for $\neg R$) that at most one contestant solved exactly 5 problems. Then, instead of (5), the best we can say is that

$$(\Sigma c\colon (\Sigma t\colon [c \text{ solved } t])) \leq 6(N-1) + 10\,,$$

since (in the situation which makes this sum as large as possible) $N-1$ contestants solved 4 problems, hence 6 pairs of problems, and one contestant solved 5 problems, hence $\binom{5}{2} = 10$ pairs of problems. Applying this inequality to (7) yields

$$(\text{Min } t\colon (\Sigma c\colon [c \text{ solved } t])) \leq \tfrac{2}{5}N + \tfrac{4}{15}\,,$$

which is weaker than the desired result. We can tighten this up a little bit by multiplying by 5:

$$5(\text{Min } t\colon (\Sigma c\colon [c \text{ solved } t])) \leq 2N + \tfrac{4}{3}\,.$$

Then, since the left-hand side is an integer,

$$5(\text{Min } t\colon (\Sigma c\colon [c \text{ solved } t])) \leq 2N + 1\,.$$

This observation lets us shave $\frac{1}{15}$ off our upper bound for the minimum; but it's still not good enough.

The $[P]$ notation is used in *Concrete Mathematics*, by Graham, Knuth, and Patashnik; they call it "Iverson's brackets", and make much use of it in evaluating sums.

The proof format used above is, apparently, due to W. H. J. Feijen; I read about it in Dijkstra. The notation for sums and whatnot resembles that of Dijkstra. Also Dijkstra's is the observation that min $\leq$ avg $\leq$ max, or rather, the observation that this is a better formulation of the pigeonhole principle; see his note EWD1094. Dijkstra also has a good discussion about choosing which of several equivalent statements one wishes to prove; see his note EWD729.

3 Other problems

The other four 2005 IMO problems (all paraphrased):

1. We are given an infinite sequence of integers $a_1, a_2, \ldots$. The sequence contains infinitely many positive values and infinitely many negative values. For every positive integer n, the values $a_1, a_2, \ldots, a_n$ leave n different remainders on division by n. Show that every integer occurs exactly once in the sequence.

2. Given positive real numbers x, y, z such that $xyz \geq 1$, show that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0 \,.$$

3. Find all positive integers relatively prime to $2^n + 3^n + 6^n - 1$ for all positive integers n. [This is a badly worded problem. I'm guessing that we are to find $\{k\colon k \in \mathbb{Z}^+ \text{ and } (\forall n \in \mathbb{Z}^+\colon\ \gcd(k, 2^n + 3^n + 6^n - 1) = 1)\}$.]

4. Given a convex quadrilateral $ABCD$ such that $BC = DA$ and BC is not parallel to DA. AC meets BD at P. E and F are variable points on BC and DA respectively such that $BE = DF$. EF meets BD at Q, and meet AC at R. Prove that the circumcircle of $\triangle PQR$ passes through a fixed point other than P. [Another badly worded problem.]