Math Club Notes: 2005 August 8

1 Rectangles

In my last email I mentioned the following problem: what integer-sided rectangles have equal area and perimeter?

Let the side lengths be x and y. We seek integer solutions to

$$xy = 2(x+y) \text{ and } x \geq 0 \text{ and } y \geq 0 . \tag{1}$$

(One usually requires lengths to be nonnegative.) Here's the slick solution:

$$\begin{aligned} xy = 2(x+y) &\iff xy - 2x - 2y = 0 \\ &\iff xy - 2x - 2y + 4 = 4 \\ &\iff (x-2)(y-2) = 4 \end{aligned}$$

Thus (1) is equivalent to

$$(x-2)(y-2) = 4 \text{ and } x - 2 \geq -2 \text{ and } y - 2 \geq -2 .$$

The integer factorizations of 4 with both factors at least -2 are $(-2,-2)$, $(2,2)$, and $(1,4)$; these give rise to solutions for (x,y) of $(0,0)$, $(4,4)$, and $(3,6)$.

The trick in this solution is, of course, to notice that $xy - 2x - 2y$ is almost factorable. In slightly more general terms, we should know that

$$xy + ax + by = (x+b)(y+a) - ab ,$$

and be ready to apply this when we see xy, x, and y terms. Call it "completing the hyperbola". (Why?)

2 Convergence of an integral

Back on July 4, we evaluated the integral

$$\int_0^{\pi} \ln \sin x \, dx = -\pi \ln 2$$

by some clever tricks. The derivation assumed that this improper integral converges; today we proved that it does.

Since the integrand is undefined at both endpoints, we must split up the integral and show the convergence at each endpoint separately. (That's how improper integrals of this type are defined.) Well, we can get away with doing just one endpoint, because the integrand is symmetric across $x = \pi/2$. And we don't need to split the integral into exactly two pieces. For any $\delta \in (0, \pi)$, we have

$$\int_0^{\pi} = \int_0^{\delta} + \int_{\delta}^{\pi-\delta} + \int_{\pi-\delta}^{\pi} .$$

On the right-hand side, the middle integral is proper, and by symmetry the first and last are equal. So it suffices to show that

$$\exists \delta \in (0, \pi) \colon \int_0^{\delta} \ln \sin x \, dx \text{ converges.} \tag{2}$$

We can choose δ in whatever way we find convenient.

Now, sometimes we show the convergence of an improper integral "directly"; for example, by definition

$$\int_0^1 \ln x \, dx \text{ converges} \iff \lim_{a \to 0^+} \int_a^1 \ln x \, dx \text{ exists} ,$$

and this latter limit-of-integral we can just evaluate:

$$\begin{aligned}
\lim_{a\to 0^+} \int_a^1 \ln x \, dx &= \lim_{a\to 0^+} \big[x \ln x - x \big]_a^1 && \text{(by parts; see July 4)} \\
&= \lim_{a\to 0^+} (1 \ln 1 - 1 - a \ln a + a) \\
&= -1 - \lim_{a\to 0^+} a \ln a \\
&= -1 - \lim_{a\to 0^+} \frac{\ln a}{1/a} \\
&= -1 - \lim_{a\to 0^+} \frac{1/a}{-1/a^2} && \text{(L'Hôpital's Rule)} \\
&= -1 - \lim_{a\to 0^+} (-a) \\
&= -1
\end{aligned}$$

Alas, this plan won't work in the case of $\int_0^\delta \ln \sin x \, dx$, because we don't know the antiderivative of $\ln \sin x$. (That's why we needed clever tricks to evaluate the definite integral.)

As usual in such problems, we now fall back on the Comparison Theorem. Since

$$\begin{aligned} 0 < x < \pi &\implies 0 < \sin x \le 1 \\ &\implies \ln \sin x \le \ln 1 \qquad \text{(ln is increasing)} \\ &\iff \ln \sin x \le 0 \end{aligned}$$

we need a function f with the following properties:

$$\forall x \in (0, \delta)\colon f(x) \le \ln \sin x \tag{3}$$

$$\int_0^\delta f(x)\, dx \text{ converges} \tag{4}$$

Then the hypotheses of the Comparison Theorem are satisfied, and (2) follows.

So now we want to find a function f and a constant $\delta \in (0, \pi)$ such that (3) and (4) hold. As usual when applying the Comparison Theorem, we will want for f something related to our integrand $\ln \sin x$ (so that (3) will be easy to establish) but simpler (so that (4) will be easy to establish).

The obvious first guess is $f(x) = \ln x$. We already know that (4) holds for this function (as shown on page 2), so all we need to show is (3). Since ln is increasing,

$$\ln x \le \ln \sin x \impliedby x \le \sin x\,,$$

but alas, the latter is false for all positive x.

For some reason, both Eileen and I had the same second guess: $f(x) = \ln x^2$. (There are alternatives; e.g. $f(x) = \ln(\frac{1}{2}x)$ also works.) Proving (4) is still easy for $\ln x^2$, since

$$\int_0^\delta \ln x^2 \, dx = \int_0^\delta 2 \ln x \, dx = 2 \int_0^\delta \ln x \, dx\,,$$

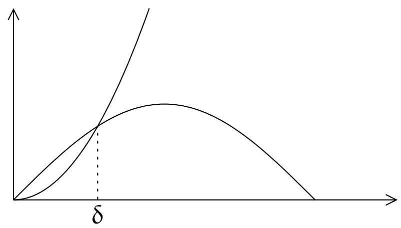
for any $\delta \ge 0$, and we know the remaining integral converges. As for (3), we want to show

$$\exists \delta \in (0, \pi)\colon \forall x \in (0, \delta)\colon\ \ln x^2 \le \ln \sin x\,,$$

which (since, again, ln is increasing) follows from

$$\exists \delta \in (0, \pi)\colon \forall x \in (0, \delta)\colon x^2 \le \sin x\,. \tag{5}$$

That there is such a δ seems obvious in a picture:



Of course, to draw this picture we are exploiting our long familiarity with these functions. How can we prove (5) more formally?

Here's one way: formalize the relevant properties of the picture. First we calculate some second derivatives to show that $\sin x$ is concave down in $(0, \pi)$ and x^2 is concave up (everywhere, but in particular) in $(0, \pi)$. Thus $\sin x$ lies above its secant lines in that interval, and x^2 lies below its secant lines in that interval. Since they intersect at $x = 0$ and $x = \delta$, as in our picture, they share the secant joining those two points of intersection, so between 0 and δ (assuming $\delta \in (0, \pi)$) we have $x^2 \leq \text{secant} \leq \sin x$, which yields (5).

Thus we need only show that these curves do actually intersect; that is, we want to find $\delta \in (0, \pi)$ such that $\delta^2 = \sin \delta$. Unfortunately, we have no techniques for solving this kind of equation; fortunately, we don't need to solve it. All we need is that such a δ exists, that is,

$$\exists \delta \in (0, \pi) \colon \delta^2 = \sin \delta \,. \tag{6}$$

Cue the Intermediate Value Theorem: Let $g(x) = x^2 - \sin x$. Then, on the one hand,

$$g(\pi) = \pi^2 - 0 > 0$$

and on the other,

$$g(\tfrac{\pi}{6}) = \frac{\pi^2}{36} - \frac{1}{2} = \frac{\pi^2 - 18}{36} < \frac{4^2 - 18}{36} < 0$$

So, knowing that $0 < \pi < 4$, we can show $g(\frac{\pi}{6}) < 0 < g(\pi)$, whence, by the continuity of g and IVT,

$$\exists \delta \in (\tfrac{\pi}{6}, \pi) \colon g(\delta) = 0 \,,$$

and (6) follows.

And that solves the problem. Huzzah!

(The solution can be considerably simplified. It makes a big difference to approach the proof of (5) from a totally different direction: First we calculate that

$$\lim_{x \to 0^+} \frac{x^2}{\sin x} = \Big(\lim_{x \to 0^+} x \Big) \Big(\lim_{x \to 0^+} \frac{x}{\sin x} \Big) = 0 \cdot 1 = 0 \,.$$

By definition this means

$$\forall\epsilon > 0\colon \exists\delta > 0\colon \forall x \in (0,\delta)\colon \left|\frac{x^2}{\sin x}\right| < \epsilon\,.$$

Taking $\epsilon = 1$ in particular, we have

$$\exists\delta > 0\colon \forall x \in (0,\delta)\colon \left|\frac{x^2}{\sin x}\right| < 1\,,$$

which can be massaged into (5). This approach is very natural if we recognize (5) as having approximately the form of the limit definition. Note how thinking pictorially led us into a more complicated solution.)

3 A hint on an outstanding problem

A hint for the outstanding problem (see June 27) of expressing

$$\sum_k \binom{n}{3k}$$

in closed form as a function on n: Supposing that we found some pattern in the values of this sum, how would we prove that the pattern continues to hold? Perhaps by induction on n. That would involve (in the inductive step) relating entries from the nth row of Pascal's triangle to entries from the $(n-1)$th row. Usually we use

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for this purpose. Applying this identity to our sum, we obtain

$$\sum_k \binom{n}{3k} = \sum_k \binom{n-1}{3k-1} + \sum_k \binom{n-1}{3k}\,.$$

The second sum on the right-hand side is what the inductive hypothesis would pertain to; the first sum, however, is new. Repeating this operation will bring in $\sum_k \binom{n}{3k-2}$ as well; repeating it a third time introduces nothing new.

Perhaps, then, we should be thinking not just about the sum with the multiples of 3, but about three sums all at once — one for each possible remainder on division by 3. To consider multiple values as one object, throw them into a vector: let

$$u_n = \begin{bmatrix} \sum_k \binom{n}{3k} \\ \sum_k \binom{n}{3k+1} \\ \sum_k \binom{n}{3k+2} \end{bmatrix}\,.$$

How is u_{n+1} related to u_n?

4 A new problem

Yet another problem from the aforementioned U of Waterloo contest: find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}$ with the property that

$$\forall x, y \in \mathbb{R}^+\colon f(x + y) = f(x^2 + y^2) \; .$$

We observed today that all constant functions have this property; a natural conjecture (upon failing to think of any others) is that these are the only such functions. ($\mathbb{R}^+$ is the set of positive real numbers.)