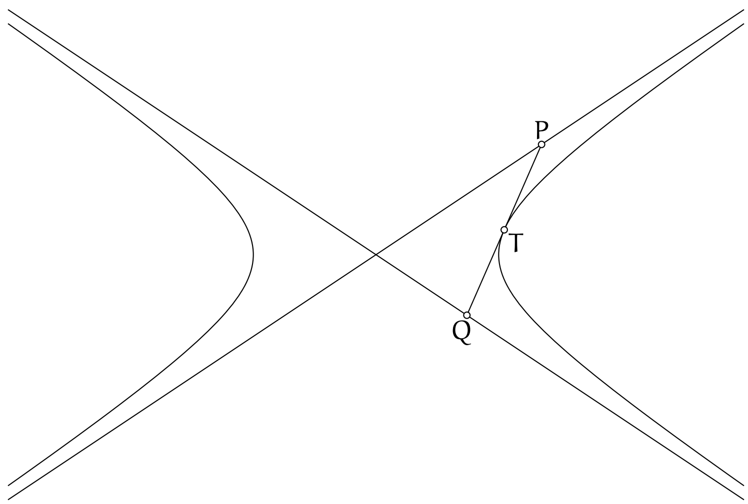
Math Club Notes: 2005 July 18

1 Hyperbola tangents

1.1 The theorem

We will prove a theorem which (according to MathWorld) is due to Apollonius: the segment of a hyperbola's tangent bounded by its asymptotes is bisected by the point of tangency. For example, in the following figure we have $PT = TQ$.



Presumably Apollonius proved this with synthetic methods. I have no idea how he did that; we'll use analytic geometry instead.

1.2 First solution

Let the hyperbola be H. (That is, H is the set of points on the hyperbola.) With a suitable coordinate system, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} \in H \iff \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,. \tag{1}$$

(A "suitable coordinate system" is one with the origin at the centre of the hyperbola and the x-axis oriented to pass through the foci.) It will be convenient later to have observed that

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right)\,.$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} \in H \iff \frac{x}{a} - \frac{y}{b} = \left(\frac{x}{a} + \frac{y}{b}\right)^{-1} \tag{2}$$

and

$$\begin{bmatrix} x \\ y \end{bmatrix} \in H \implies \frac{x}{a} - \frac{y}{b} \neq 0 \tag{3}$$

Let A_1 be the asymptote with slope b/a; since it passes through the origin,

$$\begin{bmatrix} x \\ y \end{bmatrix} \in A_1 \iff bx - ay = 0 \tag{4}$$

Implicitly differentiating (1), we obtain

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0\,, \text{ whence } y' = \frac{b^2x}{a^2y}\,,$$

that is,

$$L \text{ is tangent to } H \text{ at } (x,y) \implies L \text{ has slope } \frac{b^2x}{a^2y}\,. \tag{5}$$

So, let $T = (x_0, y_0)$ be a point on H, and let L be the tangent to H at T. This line passes through T, and its slope is given by (5); from these we obtain its point-slope equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} \in L \iff y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

A little algebra, to simplify:

$$\begin{aligned} y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0) &\iff \frac{y_0}{b^2}(y - y_0) = \frac{x_0}{a^2}(x - x_0) \\ &\iff \frac{y_0y}{b^2} - \frac{y_0^2}{b^2} = \frac{x_0x}{a^2} - \frac{x_0^2}{a^2} \\ &\iff \frac{x_0x}{a^2} - \frac{y_0y}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} \\ &\iff \frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1 \qquad \text{(by (1); } (x_0,y_0) \in H\text{)} \end{aligned}$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} \in L \iff \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1\,. \tag{6}$$

(Incidentally, note the similarity of structure between (6) and (1).)

Now let P be the intersection of the tangent L and the asymptote A_1.

$$\begin{aligned}
\overrightarrow{OP} = \begin{bmatrix} x \\ y \end{bmatrix} &\iff \begin{bmatrix} x \\ y \end{bmatrix} \in A_1 \wedge \begin{bmatrix} x \\ y \end{bmatrix} \in L \\
&\iff \begin{cases} bx - ay = 0 \\ \dfrac{x_0 x}{a^2} - \dfrac{y_0 y}{b^2} = 1 \end{cases} && \text{(by (4) and (6))} \\
&\iff \begin{bmatrix} b & -a \\ x_0/a^2 & -y_0/b^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}$$

The determinant of this matrix is $x_0/a - y_0/b$, which, by (3), is not zero; thus the matrix is invertible, and the system has a unique solution. (That is, L and A_1 are not parallel, as we'd expect.) That solution is

$$\begin{aligned}
\begin{bmatrix} b & -a \\ x_0/a^2 & -y_0/b^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{1}{x_0/a - y_0/b} \begin{bmatrix} -y_0/b^2 & a \\ -x_0/a^2 & b \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{x_0/a - y_0/b} \begin{bmatrix} a \\ b \end{bmatrix} \\
&= \left(\frac{x_0}{a} + \frac{y_0}{b} \right) \begin{bmatrix} a \\ b \end{bmatrix} && \text{(by (2))} \\
&= x_0 \begin{bmatrix} 1 \\ b/a \end{bmatrix} + y_0 \begin{bmatrix} a/b \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & a/b \\ b/a & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}
\end{aligned}$$

A similar computation shows that Q, the intersection of the tangent L with the asymptote A_2, is given by

$$\begin{bmatrix} 1 & -a/b \\ -b/a & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

and so the midpoint of PQ is

$$\begin{aligned}
&\frac{1}{2} \left(\begin{bmatrix} 1 & a/b \\ b/a & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} 1 & -a/b \\ -b/a & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right) \\
&\qquad = \frac{1}{2} \left(\begin{bmatrix} 1 & a/b \\ b/a & 1 \end{bmatrix} + \begin{bmatrix} 1 & -a/b \\ -b/a & 1 \end{bmatrix} \right) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&\qquad = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},
\end{aligned}$$

as claimed.

1.3 Second solution

Let $M\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/a & -1/b \\ 1/a & 1/b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} . \tag{7}$$

Note that, since

$$\det \begin{bmatrix} 1/a & -1/b \\ 1/a & 1/b \end{bmatrix} = \frac{2}{ab} \neq 0 ,$$

the transformation M is invertible.

Let G be the hyperbola

$$\begin{bmatrix} x \\ y \end{bmatrix} \in G \iff xy = 1 \tag{8}$$

Then:

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} \in H &\iff \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 && \text{(by (1))} \\
&\iff \left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 1 && \text{(difference of squares)} \\
&\iff \begin{bmatrix} x/a - y/b \\ x/a + y/b \end{bmatrix} \in G && \text{(by (8))} \\
&\iff \begin{bmatrix} 1/a & -1/b \\ 1/a & 1/b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in G && \text{(matrix arithmetic)} \\
&\iff M \begin{bmatrix} x \\ y \end{bmatrix} \in G && \text{(by (7))}
\end{aligned}$$

Thus G is the image of H under M.

We will prove the theorem for G, and deduce that it holds for H as well. This makes for simpler algebra than in the first solution because (8) is simpler than (1). We do, however, have to check that everything we care about in this problem is preserved by the transformation M (and its inverse M^{-1}).

One part is easy: being a midpoint is preserved by M, because (letting O denote the origin, as usual)

$$T \text{ is the midpoint of } PQ \iff \overrightarrow{OT} = \tfrac{1}{2}(\overrightarrow{OP} + \overrightarrow{OQ}) .$$

The right-hand side is a statement of vector arithmetic, so its truth or falsity is preserved by a linear transformation such as M.

The other part is whether tangency is preserved by M, that is, whether the tangent to H at T is mapped to the tangent to G at $M(T)$ (and vice versa). The argument here would be a little more complicated, but in spirit it goes something like this. If T and X are some two distinct points on H, then $M(T)$ and $M(X)$ are some two distinct points on G. (They are on G because G is the image of H; they remain distinct because M is invertible.) Since linear transformations preserve straight lines, it follows that the secant TX is mapped to the secant $M(T)M(X)$. Since linear transformations are continuous, that is, they preserve limits, the tangent at T (being the limit of secants TX) is mapped to the tangent at $M(T)$ (being the limit of secants $M(T)M(X)$).

Now to prove the theorem for G. Taking differentials in (8), we find that

$$x\,dy + y\,dx = 0\,,\text{ whence } \frac{dy}{dx} = -\frac{y}{x}\,.$$

So let L be the tangent to G at (x_0, y_0); then

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} \in L \iff & y - y_0 = -\frac{y_0}{x_0}(x - x_0) \\
\iff & \frac{y - y_0}{y_0} = \frac{x_0 - x}{x_0} \\
\iff & \frac{y}{y_0} - 1 = 1 - \frac{x}{x_0} \\
\iff & \frac{x}{x_0} + \frac{y}{y_0} = 2
\end{aligned}$$

Since the asymptotes of G are the x- and y-axes, the intersections of the tangent L with the asymptotes are its x- and y-intercepts. Setting $y = 0$ in the last equation immediately yields $x = 2x_0$, so that one of the intersection points is $(2x_0, 0)$; similarly the other is $(0, 2y_0)$. The point of tangency (x_0, y_0) is indeed halfway between these points.

Isn't that tidy?

1.4 Coordinate systems

Here's another way to look at the second solution: with a suitable coordinate system, every hyperbola has the equation $xy = 1$. Take the asymptotes as the axes, and set units so that one vertex is at $(1, 1)$.

This way of saying it feels a little strange. Normally when picking a coordinate system, we put the origin where we want, orient the axes how we want, and leave it at that. If we think of the coordinate system as fixed, this means translating the figure to a desired position, and rotating and reflecting it into a desired orientation. These are all rigid motions, so they do not disturb the geometry; the figure's coordinates change but (for example) lengths, angles, areas, collinearity, congruence, similarity, parallelism, and tangency are all unaffected. In other words, for problems concerning solely geometric properties, we may, without loss of generality, choose any position and orientation for the figure.

In this problem, however, we change coordinate systems by applying the transformation M, which is not a rigid motion. It distorts the geometry — lengths and angles change — so conclusions about the geometric properties of the figure as it looks *after* this transformation do not necessarily hold for the figure *before* this transformation.

But it's okay for this problem, because M *does* preserve the properties we care about here.

There's a well-known conception of geometry, advanced by Klein, known as the Erlanger Program, which takes transformations as fundamental. For example, rather than taking congruence of segments and congruence of angles as the fundamental concepts of our geometry, we start with a group of transformations of the plane, called "rigid motions" — translation, rotation, reflection, and compositions of these — and then define congruence as that which is preserved by these transformations. (That is, figures are said to be congruent iff they can be transformed into one another by means of a rigid motion.)

If we take a different group of transformations, we end up with a different geometry. Transformations like M belong to "affine" geometry. The second approach to the theorem of Apollonius is essentially based on the observation that it involves only properties that are preserved by affine transformations; that is, it's a theorem of affine geometry, not of Euclidean geometry.

1.5 Observation on area

It's easy to show that, for the hyperbola G, the triangle formed by the asymptotes and the tangent has area 2, no matter where the point of tangency is. Now, affine transformations do not preserve area, but they do have a predictable effect on it: they scale it by some factor, given by the determinant of the associated matrix. Since M has determinant $2/ab$, its inverse M^{-1} has determinant $ab/2$, so the asymptote-tangent triangle for H has $ab/2$ times the area of the triangle for G, that is, area ab. No matter where the point of tangency is.

(Apollonius proved that too, apparently.)