Math Club Notes: 2005 July 11

1 Continued square roots

A problem you'll find everywhere: evaluate

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}\,.$$

This an infinitely long expression, so as usual we define it in terms of limits. Define a sequence $\{a_n\}_{n=0}^{\infty}$ by the recurrence

$$\begin{aligned} a_0 &= 0 \\ a_{n+1} &= \sqrt{2+a_n} \end{aligned}$$

Thus

$$\begin{aligned} a_0 &= 0 \\ a_1 &= \sqrt{2} \\ a_2 &= \sqrt{2+\sqrt{2}} \\ a_3 &= \sqrt{2+\sqrt{2+\sqrt{2}}} \end{aligned}$$

and we can define our infinite expression by

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}} = \lim_{n\to\infty} a_n\,,$$

if this limit exists.

Suppose for the moment that the limit does exist, say,

$$\lim_{n\to\infty} a_n = L\,.$$

Then

$$\begin{aligned} L &= \lim_{n\to\infty} a_n \\ &= \lim_{n\to\infty} a_{n+1} \\ &= \lim_{n\to\infty} \sqrt{2+a_n} && \text{(by the recurrence)} \\ &= \sqrt{\lim_{n\to\infty}(2+a_n)} && \text{(square root is continuous)} \\ &= \sqrt{2+\lim_{n\to\infty} a_n} && \text{(adding 2 is continuous)} \\ &= \sqrt{2+L} \end{aligned}$$

Square both sides, solve the quadratic; either $L = -1$ or $L = 2$. The former is impossible: every a_n is at least zero — the first *is* zero, and the subsequent elements are square roots — so their limit must also be at least zero. So if the limit exists, it is 2.

How can we show that the limit exists? Since we have a pretty good guess about what it is, we could go to first principles, that is, try to prove that

$$\forall\epsilon > 0 \colon \exists N \colon n > N \implies |a_n - 2| < \epsilon \,.$$

Try this if you like; I don't see how to make it work.

Another way to show a sequence converges: show that it is increasing and bounded above. How to show that it is increasing?

$$\begin{aligned}
\{a_n\} \text{ is increasing} &\iff \forall n \colon a_n \le a_{n+1} && \text{(definition of increasing)} \\
&\iff \forall n \colon a_n \le \sqrt{2 + a_n} && \text{(by the recurrence)} \\
&\iff \forall n \colon a_n^2 \le 2 + a_n && (a_n \ge 0) \\
&\iff \forall n \colon (a_n - 2)(a_n + 1) \le 0 && \text{(algebra)} \\
&\iff \forall n \colon a_n - 2 \le 0 && (a_n + 1 \ge 1 > 0) \\
&\iff \forall n \colon a_n \le 2 && \text{(algebra)}
\end{aligned}$$

If only we could show that $\forall n \colon a_n \le 2$. That would show both that the sequence is bounded above (instantly) and that it is increasing (by the above argument).

$\forall n \colon a_n \le 2$ is a statement about all natural numbers; let's try induction. Base case: $a_0 = 0 \le 2$. Inductive step: if $a_n \le 2$ then

$$a_{n+1} = \sqrt{2 + a_n} \le \sqrt{2 + 2} = 2 \,.$$

Well, that was easy. (By the way, since our sequence is defined by a recurrence, that is, by the relationship of each element to the one before it, induction should be at the back of your mind during the whole problem. Recurrences are tailor-made for inductive arguments.)

This is an interesting example because the mere fact that the limit exists is enough to allow a computation that determines its value. A kind of bootstrapping: give an argument to show that there is a limit, then use that fact (and the properties of limits) to deduce what it is.

(The previously mentioned U of Waterloo contest, http://www.stats.uwaterloo.ca/~cgsmall/EK2001.pdf has a generalization of this problem.)

2 Examples of half-lattices

The little note on "half-lattices" attached last week has no examples of half-lattices. Here's two lattice operations on $\mathbb{N}$ (that is, the nonnegative integers):

$$a \downarrow b = \min\{a, b\}$$
$$a \downarrow b = \gcd(a, b)$$

(In order for gcd to be an operation on $\mathbb{N}$, we must define $\gcd(0, 0)$; it's best to set $\gcd(0, 0) = 0$.)

Here's a third: let $a \downarrow b$ be the number whose binary expansion has a 1 in those positions where the binary expansions of a and b both have 1s, and 0s elsewhere. For example,

$$\begin{aligned} 25 &= (11001)_2 \\ 85 &= (1010101)_2 \\ 17 &= (10001)_2 \end{aligned}$$

and so $25 \downarrow 85 = 17$. (This operation is called "bitwise and".)

I leave figuring out the associated meanings of $\sqsubseteq$ as an exercise.

3 Solving for unknowns

Remember back on June 13, when we constructed an example of an ordered ring in which $\mathbb{Z}$ was bounded above? The strategy was to assume we had such a ring (that is, a ring containing the integers and an upper bound for them, called ∞), and then to deduce other properties that that ring would have to have. Eventually we deduced enough about the ring to say what it was.

It occurred to me last week that this is the same procedure we use when solving an equation. Suppose we want to find a number which is its own square. We suppose that we have such a number, say, x; that is, we suppose that

$$x^2 = x .$$

Then with a little algebra we deduce that

$$\begin{aligned} x^2 - x &= 0 \\ x(x-1) &= 0 \end{aligned}$$

and so either $x = 0$ or $x = 1$. So: if x is a number with the desired property, then it is either 0 or 1 (and, in this instance, conversely).

It's exactly the same thing. Suppose you have an object with certain properties; use those properties to deduce others; eventually you find out enough about the object that you're willing to say you know what it is.

4 Sum of reciprocals of sums of squares

$$\begin{aligned}
\sum_{r=1}^{\infty}\left(\sum_{s=1}^{r} s^2\right)^{-1} &= \sum_{r=1}^{\infty}\frac{6}{r(r+1)(2r+1)} && \text{(well-known formula)} \\
&= \sum_{r=1}^{\infty}\left(\frac{6}{r}+\frac{6}{r+1}-\frac{24}{2r+1}\right) && \text{(partial fractions)}
\end{aligned}$$

At this point we might want to split this up into three sums:

$$\sum_{r=1}^{\infty}\frac{6}{r}+\sum_{r=1}^{\infty}\frac{6}{r+1}-\sum_{r=1}^{\infty}\frac{24}{2r+1}\,.$$

But we can't; these three sums diverge.

So instead we must start speaking of the limit of partial sums. Let

$$S_n=\sum_{r=1}^{n}\left(\frac{6}{r}+\frac{6}{r+1}-\frac{24}{2r+1}\right)\,.$$

(We are interested in determining $\lim_{n\to\infty} S_n$.) Now we can split the sum up, since the resulting three sums are finite and issues of convergence do not arise:

$$S_n=\sum_{r=1}^{n}\frac{6}{r}+\sum_{r=1}^{n}\frac{6}{r+1}-\sum_{r=1}^{n}\frac{24}{2r+1}\,.$$

What now? Well, the first two sums overlap a fair bit. Let's use that fact:

$$\begin{aligned}
S_n &= \sum_{r=1}^{n}\frac{6}{r}+\sum_{r=2}^{n+1}\frac{6}{r}-\sum_{r=1}^{n}\frac{24}{2r+1} \\
&= 6+\frac{6}{n+1}+\sum_{r=2}^{n}\frac{6}{r}+\sum_{r=2}^{n}\frac{6}{r}-\sum_{r=1}^{n}\frac{24}{2r+1} \\
&= 6+\frac{6}{n+1}+\sum_{r=2}^{n}\frac{12}{r}-\sum_{r=1}^{n}\frac{24}{2r+1}
\end{aligned}$$

Now, observe that the second sum has consecutive odd numbers in the denominators; where are the even numbers? What would a sum with the even numbers look like? It would have $2r$ in the denominator. Well, we can make that happen:

$$S_n=6+\frac{6}{n+1}+\sum_{r=2}^{n}\frac{24}{2r}-\sum_{r=1}^{n}\frac{24}{2r+1}$$

(You could also reach this point by trying to turn the 12 into 24 to match the other sum.) Now we have even numbers and odd numbers in two sums. Let's write it to express that:

$$S_n = 6 + \frac{6}{n+1} + \sum_{\substack{4 \le k \le 2n \\ k \text{ even}}} \frac{24}{k} - \sum_{\substack{3 \le k \le 2n+1 \\ k \text{ odd}}} \frac{24}{k}$$

Now the summands are identical, and their ranges interlace nicely to form all numbers from 3 to $2n+1$. Well, the summands are actually not quite identical: the ones with even k are positive and the ones with odd k are negative. But we know how to write that:

$$S_n = 6 + \frac{6}{n+1} + \sum_{k=3}^{2n+1} \frac{24(-1)^k}{k}$$

At this point we must remember having seen a sum like this before. If we do, it's easy to proceed:

$$\begin{aligned}
S_n &= 6 + \frac{6}{n+1} + 24 \sum_{k=3}^{2n+1} \frac{(-1)^k}{k} \\
&= 6 + \frac{6}{n+1} + 24 \left(1 - \frac{1}{2} + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right) \\
&= 18 + \frac{6}{n+1} + 24 \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \\
&= 18 + \frac{6}{n+1} - 24 \sum_{k=0}^{2n} \frac{(-1)^k}{k+1}
\end{aligned}$$

We don't have a closed form for S_n here, but (see our notes for June 13) we know the limit of the remaining sum as $n \to \infty$. Final result:

$$\frac{1}{1} + \frac{1}{1+4} + \frac{1}{1+4+9} + \cdots = 18 - 24 \ln 2\,.$$