Math Club Notes: 2005 July 4

1 An integral

The notes for June 13 mention the problem of evaluating

$$\int_0^{\pi} \ln \sin x \, dx \,.$$

(This problem is from http://www.stats.uwaterloo.ca/~cgsmall/EK2001.pdf, an old competition for undergrads at the University of Waterloo.[1] Same source as the sum-of-products-of-subsets problem discussed in the notes for June 20.)

Mehran suggested that we consider how we'd proceed in the simpler related problem of evaluating

$$\int \ln x \, dx \,.$$

It turns out that this integral is done by parts, with $u = \ln x$ and $dv = dx$, which yields

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - x + C \,.$$

Trying "the same thing" in our problem (that is, integrating by parts with $u = \ln \sin x$ and $dv = dx$) yields

$$\int \ln \sin x \, dx = x \ln \sin x - \int x \cot x \, dx \,.$$

Unfortunately, the new integral is not much more appealing than the one we already had. (It's a good idea — trying something that works in a similar problem — but it doesn't always work out.)

Eileen, seeing the composition $\ln \sin x$, suggested the substitution $u = \sin x$. (Another good idea that usually works.) To make this substitution we'll need to create $du = \cos x \, dx$, so:

$$\begin{aligned} \int \ln \sin x \, dx &= \int \frac{\ln \sin x}{\cos x} \cdot \cos x \, dx \\ &= \int \frac{\ln \sin x}{\sqrt{1 - \sin^2 x}} \, d(\sin x) \\ &= \int \frac{\ln u}{\sqrt{1 - u^2}} \, du \end{aligned}$$

(Well, we have to watch out for the bounds; $\cos x = \sqrt{1 - \sin^2 x}$ is not true throughout the interval $[0, \pi]$.)

[1] I later found this problem in C.V. Durell and A. Robson, *Advanced Trigonometry* (London: G. Bell and Sons, 1930; New York: Dover, 2003), 76. Problem A5 from the 2005 Putnam (see our notes for December 8) occurs on the same page.

Not much simpler. And normally, when we see $\sqrt{1-u^2}$, we'd do the trigonometric substitution $u = \sin x$... and get back where we started.

So far, no progress.

What these approaches have in common is this: we're trying to evaluate the indefinite integral first, expecting to plug in the definite integral's bounds later. That's what we usually do, after all.

But there are situations where we can evaluate a definite integral without antidifferentiating. For example:

$$\int_{-1}^{1} \frac{\arctan x}{\sqrt{2+\sin(x^2)}}\,dx = 0$$

Rather than try to figure out the antiderivative of this hideous thing, we just note that the integrand is odd (as the quotient of an odd and an even function) and the interval of integration is symmetric about zero. Therefore the integral is zero.

Maybe we can do something similar in our problem, maybe also by exploiting symmetry. Our integrand is not odd or even, but back when we first glanced at the integral, Ray noticed that, since $\sin x$ is symmetric across $x = \pi/2$, our integrand $\ln \sin x$ is too. That is:

$$\int_0^{\pi} \ln \sin x\,dx = 2\int_0^{\pi/2} \ln \sin x\,dx\,.$$

Still doesn't seem like much progress. But here's another notion: maybe we can relate our integral to this new integral in some other way. (Just like when perturbing a sum (see our notes for May 9), or, for that matter, when integrating by parts a few times and getting back to where we started.)

The only difference between our old integral and the new integral is the bounds. How can we change the bounds of an integral? Substitution does that. What substitution turns $[0, \pi]$ into $[0, \frac{\pi}{2}]$? Let's try $u = \frac{1}{2}x$.

$$\int_0^{\pi} \ln \sin x\,dx = 2\int_0^{\pi/2} \ln \sin 2u\,du$$

It's not hard to see where to go from here.

$$\begin{aligned}
&= 2\int_0^{\pi/2} \ln(2\sin u \cos u)\,du \\
&= 2\int_0^{\pi/2} (\ln 2 + \ln \sin u + \ln \cos u)\,du \\
&= 2\int_0^{\pi/2} \ln 2\,du + 2\int_0^{\pi/2} \ln \sin u\,du + 2\int_0^{\pi/2} \ln \cos u\,du
\end{aligned}$$

The first integral is trivial. The second is what we were trying to make. The third isn't, but by considering the various relationships we know between sin and cos, we notice that

$$\begin{aligned}\int_0^{\pi/2} \ln\cos u\,du &= \int_0^{\pi/2} \ln\sin(u+\tfrac{\pi}{2})\,du \\ &= \int_{\pi/2}^{\pi} \ln\sin v\,dv && (v = u+\tfrac{\pi}{2}) \\ &= \int_0^{\pi/2} \ln\sin v\,dv && \text{(symmetry again)}\end{aligned}$$

Huzzah! Substitute this back, and our original integral $\int_0^\pi$ has been expressed in terms of $\int_0^{\pi/2}$ in another way, and just as desired, this enables us to solve for the original integral.

Putting it all together in a tidy way:

$$\begin{aligned}\int_0^\pi \ln\sin x\,dx &= 2\int_0^{\pi/2} \ln\sin 2u\,du && (u = \tfrac{1}{2}x) \\ &= 2\int_0^{\pi/2} \ln(2\sin u\cos u)\,du \\ &= 2\int_0^{\pi/2} (\ln 2 + \ln\sin u + \ln\cos u)\,du \\ &= \pi\ln 2 + 2\int_0^{\pi/2} \ln\sin u\,du + 2\int_0^{\pi/2} \ln\cos u\,du \\ &= \pi\ln 2 + 2\int_0^{\pi/2} \ln\sin u\,du + 2\int_0^{\pi/2} \ln\sin(u+\tfrac{\pi}{2})\,du \\ &= \pi\ln 2 + 2\int_0^{\pi/2} \ln\sin u\,du + 2\int_{\pi/2}^{\pi} \ln\sin v\,dv && (v = u+\tfrac{\pi}{2}) \\ &= \pi\ln 2 + 2\int_0^{\pi} \ln\sin u\,du\end{aligned}$$

and solving for our integral yields

$$\int_0^\pi \ln\sin x\,dx = -\pi\ln 2\,.$$

(In this "tidy" derivation, where did we use symmetry? And, by the way: does it matter that it's ln instead of, say, $\log_7$?)

A loose end: The integral is improper, and we have assumed throughout that it converges, so that the usual manipulations apply. Can you prove that it converges?

What about, say,

$$\int_0^{\pi/2} \ln\tan x\,dx?$$

2 A sum

Ray gave us his solution of a sum from June 20, namely

$$\sum_{r=1}^{\infty}\left(\sum_{s=1}^{r} s\right)^{-1} = \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots$$

We recall that

$$1+2+3+\cdots+r = \frac{r(r+1)}{2},$$

so our sum is

$$\sum_{r=1}^{\infty} \frac{2}{r(r+1)}.$$

Factoring out the 2 and looking at the first few partial sums, we find that

$$\begin{aligned}
2\sum_{r=1}^{1} \frac{1}{r(r+1)} &= 2\left(\frac{1}{1\cdot 2}\right) = 2\left(\frac{1}{2}\right) \\
2\sum_{r=1}^{2} \frac{1}{r(r+1)} &= 2\left(\frac{1}{2} + \frac{1}{2\cdot 3}\right) = 2\left(\frac{2}{3}\right) \\
2\sum_{r=1}^{3} \frac{1}{r(r+1)} &= 2\left(\frac{2}{3} + \frac{1}{3\cdot 4}\right) = 2\left(\frac{3}{4}\right)
\end{aligned}$$

and it sure looks like

$$\sum_{r=1}^{n} \frac{2}{r(r+1)} = \frac{2n}{n+1}.$$

This is easily shown to be correct by induction: we have the base case above; the inductive step is

$$\begin{aligned}
\sum_{r=1}^{n+1} \frac{2}{r(r+1)} &= \sum_{r=1}^{n} \frac{2}{r(r+1)} + \frac{2}{(n+1)(n+2)} \\
&= \frac{2n}{n+1} + \frac{2}{(n+1)(n+2)} \\
&= \frac{2n(n+2)+2}{(n+1)(n+2)} \\
&= \frac{2(n^2+2n+1)}{(n+1)(n+2)} \\
&= \frac{2(n+1)}{n+2}
\end{aligned}$$

The infinite sum (i.e., the limit as $n \to \infty$) is, then, 2.

This method — look at the first few values, guess the pattern, prove it by induction — is perfectly good. Alas, as mentioned last week in connection with $\sum_k \binom{n}{3k}$, we can't always guess the pattern; Ray reports that the pattern of the partial sums for the second sum from June 20 is not obvious.

It would be nice, then, if we could solve these problems by derivation rather than guessing. Here's one way: recall that, if we were integrating $2/r(r+1)$ instead of summing it, we'd use partial fractions. Try it here:

$$\sum_{r=1}^{n} \frac{2}{r(r+1)} = \sum_{r=1}^{n} \left(\frac{2}{r} - \frac{2}{r+1} \right)$$

Hey — a telescoping sum! (See the notes for May 22.) Thus

$$\sum_{r=1}^{n} \frac{2}{r(r+1)} = 2 - \frac{2}{n+1} ,$$

which is equivalent to Ray's conclusion above.

Maybe partial fractions would help with the second sum from June 20, eh?

3 Outstanding problems

For reference, our outstanding problems:

1. (June 13) Complete today's evaluation of

   $$\int_0^{\pi} \ln \sin x \, dx$$

   by showing that this improper integral converges.

2. (June 20) Evaluate

   $$\sum_{r=1}^{\infty} \left(\sum_{s=1}^{r} s^2 \right)^{-1} = \frac{1}{1} + \frac{1}{1+4} + \frac{1}{1+4+9} + \frac{1}{1+4+9+16} + \cdots .$$

3. (June 27) Evaluate

   $$\sum_k \binom{n}{3k} .$$

   (Here "evaluate" means "express in closed form as a function of n".)