Math Club Notes: 2005 June 20

1 Bitstrings and ants

Another tidbit from my Cmpt 272 course last winter: how many bitstrings of length n are there with no consecutive zeroes? (A bitstring is a finite sequence of 0s and 1s.)

Perhaps it is not obvious how to proceed in this question. There is a saying among programmers: when in doubt, use brute force. Let's just count all such bitstrings for the first few n.

n	Bitstrings								Count
1	0	1							2
2	01	10	11						3
3	010	011	101	110	111				5
4	0101	0110	0111	1010	1011	1101	1110	1111	8

Recognize those numbers?

If this sequence is what it appears to be, then we should be able to explain, for example, the 8 bitstrings of length 4 as being $3 + 5$ bitstrings, somehow related to the 3 bitstrings of length 2 and the 5 bitstrings of length 3. After staring at the bitstrings of length 4 for a while, we see how to make that work. Three of them are 01 followed by one of the bitstrings of length 2:

$$01\underline{01} \qquad 01\underline{10} \qquad 01\underline{11}$$

The other five are 1 followed by one of the bitstrings of length 3:

$$1\underline{010} \qquad 1\underline{011} \qquad 1\underline{101} \qquad 1\underline{110} \qquad 1\underline{111}$$

It's easy to see that a bitstring of any other form would have two consecutive zeroes. Writing this idea out a bit more formally:

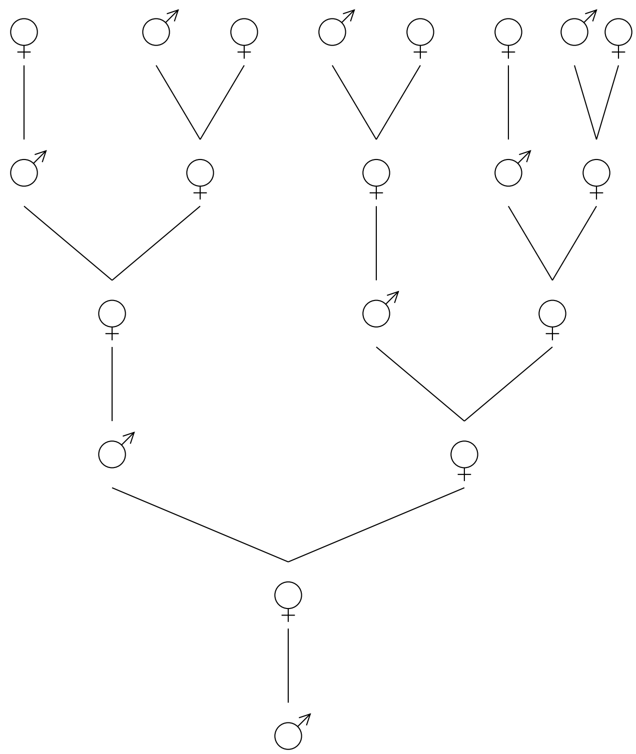
Let s_n denote the number of bitstrings of length n with no consecutive zeroes. Evidently $s_0 = 1$ (since the bitstring of length 0, namely the empty string, has no consecutive zeroes), and $s_1 = 2$ (since neither bitstring of length 1 has consecutive zeroes). If $n \geq 2$, then count the bitstrings of length n as follows. The first bit of the string must be either 0 or 1. If it is 1, the remaining bits then form one of the s_{n-1} bitstrings of length $n - 1$. If the first bit is 0, then to avoid consecutive zeroes the second bit must be 1, and the remaining bits then form one of the s_{n-2} bitstrings of length $n - 2$. Thus s_n satisfies the recurrence

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 2 \\ s_n &= s_{n-1} + s_{n-2} \qquad (n \geq 2) \end{aligned}$$

and a trivial induction shows that $s_n = F_{n+2}$, the $(n+2)$th Fibonacci number.

Many weeks after seeing the above in class, I finally noticed that we can also relate these bitstrings directly to the original Fibonacci problem about rabbits. Well, that problem sucks. Let's do insects instead.

In some species of insect — such as ants, I think — males are produced parthenogenetically by females, but females are produced by the more familiar type of mating. That is, a male has just a mother, no father, while a female has both a mother and a father. Consider the family tree of a male of such a species:



It's easy to see that this is the same structure as in Fibonacci's rabbit problem, just upside down, and with "male ant" and "female ant" instead of "pair of young rabbits" and "pair of mature rabbits". Thus the number of ancestors in generation n (counting the male ant at the bottom as generation 1) is F_n.

The bitstrings with no consecutive zeroes appear in this tree as lineage traces. In any path through the tree starting at the bottom and ending at the top, there will be no consecutive male ants, since males do not have fathers.

2 Distributivity

The notes for May 30 mention that distributivity can be understood as meaning that, for example, when

$$(a_1 + a_2)(b_1 + b_2 + b_3)(c_1 + c_2 + c_3)$$

is multiplied out, the result is a sum of products, one for every combination of one a_i, one b_i, and one c_i. You can read a product such as the above to emphasize this: where you have $+$, say "one of"; where you have $\cdot$, say "and". In this example, you'd say:

> One of a_1, a_2,
> and one of b_1, b_2, b_3,
> and one of c_1, c_2, c_3.

This describes how to make the terms in the full expansion.

Today we looked at a couple applications of this fact.

2.1 Sum of products of subsets

A problem from a contest for undergrads at the University of Waterloo:

> Prove that
> $$\sum \frac{1}{i_1 i_2 \cdots i_k} = 2001$$
> where the summation is over all nonempty subsets $\{i_1, i_2, \ldots, i_k\}$ of the set $\{1, 2, \ldots, 2001\}$.

(The contest was held in 2001.)

The number 2001 is awkwardly large. To get a handle on what the problem is talking about, let's use 4 instead. Then we wish to sum "over all nonempty subsets" of $\{1, 2, 3, 4\}$; these are

$\{1\}$	$\{3\}$	$\{1,2\}$	$\{1,4\}$	$\{2,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,2,3,4\}$
$\{2\}$	$\{4\}$	$\{1,3\}$	$\{2,3\}$	$\{3,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	

Now, a linguistic note: a phrase such as

> … all nonempty subsets $\{i_1, i_2, \ldots, i_k\}$ …

is an example of what grammarians call "apposition". It's like saying, "my brother-in-law, the idiot"; the two noun phrases "my brother-in-law" and "the idiot" refer to the same person, and this is indicated by just sticking them together. Similarly, in the problem, the words "all nonempty subsets" and the symbols "$\{i_1, i_2, \ldots, i_k\}$" refer to the same thing. Mathspeak uses apposition this way to assign names. That is, here we are to understand that $\{i_1, i_2, \ldots, i_k\}$ is a name for a nonempty subset, in fact, a name for all of them, varying during the sum. So, for example, for one of the terms we take $\{i_1, i_2, \ldots, i_k\} = \{1, 3, 4\}$, so that in this instance $k = 3$, and, say, $i_1 = 1$, $i_2 = 3$, and $i_3 = 4$, and the term corresponding to this subset is $1/(1 \cdot 3 \cdot 4)$.

I describe all this in detail because it is rarely explained, and because in this particular problem the author's notation is annoyingly difficult. I'd say it this way: Let $A = \{1, 2, \ldots, 2001\}$. Prove that

$$\sum_{\substack{S \subseteq A \\ S \neq \varnothing}} \prod_{k \in S} \frac{1}{k} = 2001 .$$

Anyway, the sum we are dealing with (using 4 instead of 2001) is

$$\begin{aligned} &\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\ &+ \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 4} \\ &+ \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 4} + \frac{1}{1 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4} \\ &+ \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \end{aligned}$$

You might wish to verify that these do indeed add up to 4.

This problem can be solved easily by applying a special case of our distributivity trick. Consider this example:

$$(1 + a)(1 + b)(1 + c) = 1 + a + b + c + ab + ac + bc + abc$$

Reading the left-hand side as suggested above, we start with "one of 1, a", or, a bit more naturally, "either 1 or a". But since taking 1 does nothing to the resulting product, it's even more natural to say "take a, or don't". So the eight terms on the right-hand side arise from choosing variously whether to take a or not, whether to take b or not, and whether to take c or not. Thus we get one term for every subset of $\{a, b, c\}$. The empty set gives rise to the term 1.

Now, the problem wants a sum with one term for every *nonempty* subset; call that N. Then $1 + N$ is the sum with one term for *every* subset, and our distributivity trick applies:

$$\begin{aligned} 1 + N &= \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{2001}\right) \\ &= \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{2002}{2001} \end{aligned}$$

Much cancellation — this is a telescoping product. So

$$1 + N = \frac{2002}{1}$$

and we're done.

2.2 Sum of divisors

Let $\sigma(n)$ denote the sum of the (positive) divisors of n. For example,

$$\sigma(60) = 1 + 2 + 3 + 4 + 5 + 6 + 10 + 12 + 15 + 20 + 30 + 60 = 168\,.$$

If we happen to know the prime factorization of n, we can assemble its divisors easily. The divisors of $60 = 2^2 \cdot 3 \cdot 5$, for example, can have only 2, 3, and 5 in their prime factorizations, and with exponents that are not greater than the corresponding exponents in the prime factorization of 60. Thus the divisors of 60 are

$$\begin{array}{lll} 2^0 3^0 5^0 = 1 & 2^1 3^0 5^0 = 2 & 2^2 3^0 5^0 = 4 \\ 2^0 3^1 5^0 = 3 & 2^1 3^1 5^0 = 6 & 2^2 3^1 5^0 = 12 \\ 2^0 3^0 5^1 = 5 & 2^1 3^0 5^1 = 10 & 2^2 3^0 5^1 = 20 \\ 2^0 3^1 5^1 = 15 & 2^1 3^1 5^1 = 30 & 2^2 3^1 5^1 = 60 \end{array}$$

Their sum, then, is a sum of products, each consisting of

one of $2^0, 2^1, 2^2$,
and one of $3^0, 3^1$,
and one of $5^0, 5^1$.

Thus

$$\sigma(60) = (1 + 2 + 4)(1 + 3)(1 + 5) = 7 \cdot 4 \cdot 6 = 168\,.$$

In general,

$$\sigma(2^{e_2} 3^{e_3} 5^{e_5} \cdots) = \prod_{p \text{ prime}} (1 + p + p^2 + \cdots + p^{e_p})\,,$$

where e_p is a nonnegative integer for every prime p. Of course, that's a geometric series in there, so we can also write this as

$$\sigma(2^{e_2} 3^{e_3} 5^{e_5} \cdots) = \prod_{p \text{ prime}} \frac{p^{e_p+1} - 1}{p - 1}\,.$$

(This result is well-known; it appears in all introductions to number theory. I've never seen it explained quite this way, though.)

3 Sums of reciprocals of sums

A couple sums for y'all to think about:

$$\sum_{r=1}^{\infty} \left(\sum_{s=1}^{r} s \right)^{-1} = \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots$$
$$\sum_{r=1}^{\infty} \left(\sum_{s=1}^{r} s^2 \right)^{-1} = \frac{1}{1} + \frac{1}{1+4} + \frac{1}{1+4+9} + \frac{1}{1+4+9+16} + \cdots$$

The problem is, of course, to evaluate them. The first one is easier. You might find some bits of the notes for May 9, May 22, and June 13 useful.