Math Club Notes: 2005 May 9

1 Implication

We discussed the fact that $p \to q$ is true whenever p is false. This is a common difficulty when first introduced to symbolic logic.

One way to think of it: It's not so much that $p \to q$ is right when p is false, it's just that it's not wrong. (Suppose I say, "If it rains today, then I will take my umbrella.". Suppose that then it doesn't rain. I didn't say anything about that situation, so I can't very well be wrong, can I?)

Another: Suppose we decided that $p \to q$ would be false when p is false. Then the truth table for $p \to q$ would be the same as for $p \wedge q$ (i.e., "p and q"). But surely they don't mean the same thing. (Are there other options for the truth table? What's wrong with them?)

2 Sums

We discussed Ray's question: finding a closed form for

$$S_n = 1 + x + x^2 + \cdots + x^{n-1} .$$

Adding in the next term, we get

$$\begin{aligned} S_n + x^n &= 1 + x + x^2 + \cdots + x^{n-1} + x^n \\ &= 1 + x(1 + x + \cdots + x^{n-2} + x^{n-1}) \\ &= 1 + xS_n . \end{aligned}$$

Now solving for S_n yields

$$S_n = \frac{x^n - 1}{x - 1} ,$$

which is the desired closed form.

This is often a successful approach with sums. One of my books calls it "perturbing the sum". The general plan is: add the next term in the sum, then try to get from the result back to the original sum by another path. Then you have an equation in S_n; solve for it.

Another example: find a closed form for

$$S_n = 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 8 + \cdots + n2^n = \sum_{k=1}^{n} k2^k .$$

Adding in the next term, we get

$$S_n + (n+1)2^{n+1} = \sum_{k=1}^{n+1} k2^k = \sum_{k=0}^{n} (k+1)2^{k+1} = \sum_{k=0}^{n} k2^{k+1} + \sum_{k=0}^{n} 2^{k+1}$$

Now, in the first sum, the $k = 0$ term is zero, so we can just drop it and obtain

$$\sum_{k=0}^{n} k2^{k+1} = \sum_{k=1}^{n} k2^{k+1} = 2\sum_{k=1}^{n} k2^{k} = 2S_n \, .$$

The second sum is just a geometric series, which we solved above:

$$\sum_{k=0}^{n} 2^{k+1} = 2\sum_{k=0}^{n} 2^{k} = 2 \cdot \frac{2^{n+1}-1}{2-1} = 2^{n+2} - 2 \, .$$

Putting it together and solving for S_n yields

$$S_n = (n-1)2^{n+1} + 2 \, .$$

Exercise: $\sum_{k=0}^{n} F_n$, where F_n is the nth Fibonacci number. ($F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$.)

An example of how this method can fail: let

$$\Box_n = 1 + 4 + 9 + \cdots + n^2 = \sum_{k=1}^{n} k^2 \, .$$

Following the method, we add in the next term and obtain

$$\begin{aligned} \Box_n + (n+1)^2 &= \sum_{k=1}^{n+1} k^2 = \sum_{k=0}^{n} (k+1)^2 = \sum_{k=0}^{n} (k^2 + 2k + 1) \\ &= \sum_{k=0}^{n} k^2 + 2\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 \\ &= \Box_n + 2\sum_{k=0}^{n} k + (n+1) \, . \end{aligned}$$

Alas, now the $\Box_n$ terms cancel each other. But from this point we *can* solve for the remaining sum:

$$\sum_{k=0}^{n} k = \tfrac{1}{2} n(n+1) \, .$$

So this experience produces an idea for evaluating $\Box_n$: apply the perturbation method to the sum of the first n cubes, have that sum cancel itself out and leave us with an equation involving only sums of lower powers. Try it.

3 Order

We discussed the idea of an ordered field, which (in Shilov's treatment) is a field together with a binary relation $\leq$ satisfying the following axioms for all field elements x, y, and z:

1. $x \leq x$.
2. Either $x \leq y$ or $y \leq x$.
3. If $x \leq y$ and $y \leq x$, then $x = y$.
4. If $x \leq y$ and $y \leq z$, then $x \leq z$.
5. If $x \leq y$, then $x + z \leq y + z$.
6. If $0 \leq x$ and $0 \leq y$, then $0 \leq xy$.

The first four state that $\leq$ is what they call a *total order*. (A *partial order* is a relation that satisfies 1, 3 and 4, but not necessarily 2. Exercise: show that "divides" is a partial order on $\mathbb{Z}^+$.) The fifth and sixth state that $\leq$ is, in a certain sense, compatible with the arithmetic operations of the field.

Evidently $\mathbb{R}$ and its subfields (e.g., $\mathbb{Q}$, $\mathbb{Q}[\sqrt{5}]$) are ordered fields; the usual "less than or equal to" relation satisfies all the above axioms.

Shilov goes on to define $\geq$, $<$, and $>$ in the obvious way, and proves a bunch of elementary results about inequalities. Basically this is to show that these axioms are sufficient to re-create the algebra of inequalities that we all know and love. For example:

Theorem 1 $x \leq y$ if and only if $-y \leq -x$.

Proof If $x \leq y$, then by axiom 5 (with $z = -x - y$), we have $x - x - y \leq y - x - y$, that is, $-y \leq -x$. Conversely, if $-y \leq -x$, adding $x + y$ to both sides yields $x \leq y$. □

That is, multiplying an inequality by -1 reverses it. Another example:

Theorem 2 $0 \leq x^2$.

Proof By axiom 2, either $0 \leq x$ or $x \leq 0$. If $0 \leq x$, then by axiom 6, we have $0 \leq xx = x^2$. If, on the other hand, $x \leq 0$, then by the previous theorem, $0 \leq -x$, and so, by axiom 6 again, $0 \leq (-x)(-x) = (-x)^2 = x^2$. □

This relies on the fact that $x^2 = (-x)^2$, which holds in any ring, hence in any field. (Proof?)

Such results lead to more interesting ones. For example, $\mathbb{C}$ is not an ordered field, that is, there is no relation on $\mathbb{C}$ satisfying the given axioms. Proof: Suppose there were. Then we'd have $0 \leq i^2 = -1$ by the second theorem above, and so $1 \leq 0$ by the first. But also $0 \leq 1^2 = 1$. So, by axiom 3, we have $0 = 1$, which is false in $\mathbb{C}$.

Exercise: Prove that $\mathbb{Z}_p$ is not an ordered field.

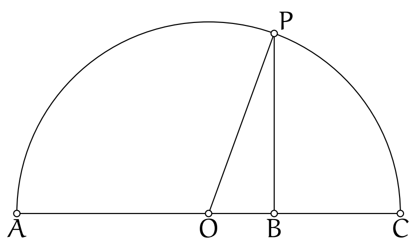
4 Inequality

We looked at the arithmetic-geometric mean inequality, which states that, for any positive reals $x_1, x_2, \ldots, x_n$,

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \tfrac{1}{n}(x_1 + x_2 + \cdots + x_n)\ ,$$

and equality is attained exactly when all the x_i are equal. The case $n = 2$, at least, was known in antiquity, and can be proved in a few fairly quick ways. Here's one: note that $(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0$; expand the left-hand side and rearrange.

Another: Let $x_1 = AB$ and $x_2 = BC$, with B between A and C. Bisect AC at O, and describe a semicircle on AC as a diameter. Erect BP perpendicular to AC, intersecting the semicircle at P.



Then $OP = \frac{1}{2}(x_1 + x_2)$ and $BP = \sqrt{x_1 x_2}$. (Why?) If $x_1 = x_2$, then B and O coincide and $OP = BP$. If $x_1 \neq x_2$, then OPB is a right triangle, and its hypotenuse is its longest side, that is, $OP > BP$.

Alas, these methods do not generalize to $n > 2$. For the general case, see Dijkstra's graceful presentation of a well-known proof: http://www.cs.utexas.edu/users/EWD/ewd11xx/EWD1140.PDF. My favourite by far, and the one we looked at on Monday. (It can be considered an induction on the number of the x_i which differ from the average of the x_i.)

This inequality is often handy when you're trying to maximize a product, keeping the sum constant, or, conversely, minimize a sum, keeping the product constant. For example, consider this routine calculus problem: what is the largest possible area of a rectangle whose perimeter is p?

Let the side lengths be x and y. We wish to maximize xy, subject to the constraint $x + y = p/2$. By the arithmetic-geometric mean inequality,

$$xy = (\sqrt{xy})^2 \leq \left(\frac{x+y}{2}\right)^2 = \left(\frac{p}{4}\right)^2\ ,$$

and equality is attained exactly when $x = y$. Thus the maximum possible area is $(p/4)^2$, and it is attained by the square.

Isn't that nicer than using calculus?

Exercise: What is the minimum possible surface area of a rectangular solid whose volume is 1, and for what solid is that minimum attained?