Thales' theorem and completing the square

The identity

$$(x-p)(x-q) = \left(x - \frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2 .$$

can be verified by factoring the right-hand side as a difference of squares, then simplifying the factors, or by expanding the left-hand side and completing the square.

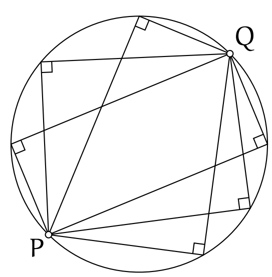
These manipulations rely essentially only on the commutativity of multiplication and its distributivity over addition. Real inner products are also commutative, and distribute over (vector) addition, so the same manipulations establish the identity

$$\langle \vec{x}-\vec{p}, \vec{x}-\vec{q} \rangle = \|\vec{x} - \tfrac{1}{2}(\vec{p}+\vec{q})\|^2 - \|\tfrac{1}{2}(\vec{p}-\vec{q})\|^2$$

(where $\langle \cdot, \cdot \rangle$ is a real inner product and $\|\cdot\|$ is the norm it induces), from which it immediately follows that

$$\langle \vec{x}-\vec{p}, \vec{x}-\vec{q} \rangle = 0 \iff \|\vec{x} - \tfrac{1}{2}(\vec{p}+\vec{q})\| = \|\tfrac{1}{2}(\vec{p}-\vec{q})\| .$$

The left-hand side of this equivalence states that the lines joining point $\vec{x}$ to points $\vec{p}$ and $\vec{q}$ are orthogonal. The right-hand side states that $\vec{x}$ lies on a sphere centred at $\frac{1}{2}(\vec{p}+\vec{q})$ and of radius $\|\frac{1}{2}(\vec{p}-\vec{q})\|$, which is exactly the sphere with the line segment joining $\vec{p}$ and $\vec{q}$ as a diameter.



This equivalence, in other words, tells us that, given points P and Q, the locus of points X for which $\angle PXQ = 90°$ is the sphere with PQ as diameter. It is Thales' theorem (plus converse) for any real inner product space.