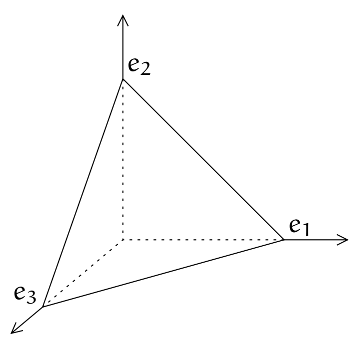
The largest centred ball in a simplex

(This note was produced as part of a research project in convex geometry under the supervision of Dr. Alexander Litvak, and funded by an NSERC Undergraduate Student Research Award.)

Definition A *unit* n-simplex is an n-simplex whose vertices $(u_i)_{i=1}^{n+1}$ satisfy

$$\langle u_i, u_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ -\frac{1}{n} & \text{if } i \neq j. \end{cases}$$

An example of a unit simplex can be constructed from the convex hull of the standard basis vectors in $\mathbb{R}^{n+1}$. Here, for example, is the equilateral triangle (a regular 2-simplex) thus formed in $\mathbb{R}^3$:



Translating this figure to put its centre at the origin, we obtain a simplex with vertices $(v_i)_{i=1}^{n+1}$, where

$$v_i = e_i - \frac{1}{n+1} \sum_{k=1}^{n+1} e_k \, .$$

To normalize, we set $u_i = \frac{1}{|v_i|} v_i$. It is easy to calculate that these points satisfy the condition stated in the definition above.

(As this example suggests, the unit simplexes are precisely those regular simplexes that are centred at the origin and have radius 1; the proof of this fact is omitted since it is not relevant to what follows.)

It is easy to show that a unit simplex contains $\frac{1}{n} B_2^n$, the Euclidean ball centred at the origin and of radius $\frac{1}{n}$. Indeed, from the definition we see that the facet opposite u_i, that is, the convex hull of all the other vertices, lies in the hyperplane $\{x \colon \langle x, u_i \rangle = -\frac{1}{n}\}$. This hyperplane is one of the supporting hyperplanes of $\frac{1}{n} B_2^n$; since the unit simplex is the intersection of the associated halfspaces, the unit simplex contains $\frac{1}{n}B_2^n$. In fact, as the following proposition states, this is in a sense the largest ball that a simplex can contain. (The proposition is an easy consequence of John's theorem; the proof given here is elementary.)

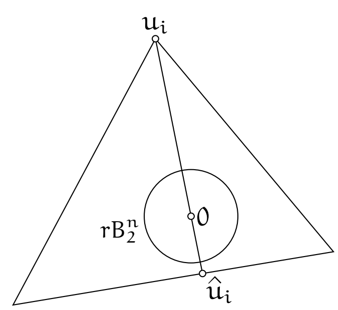
Proposition Let $(u_i)_{i=1}^{n+1}$ be points in $\mathbb{R}^n$ such that

$$rB_2^n \subseteq \operatorname{conv}\{u_i\} \subseteq B_2^n ,$$

where $r > 0$. Then $r \le \frac{1}{n}$, and equality is attained just when the u_i are the vertices of a unit simplex.

Proof Since $rB_2^n \subseteq \operatorname{conv}\{u_i\}$, the u_i are affinely independent. Thus they are the vertices of an n-simplex.

For each i, let $\hat{u}_i$ be the point where the line through the origin and u_i intersects the facet opposite u_i.



Since $rB_2^n \subseteq \operatorname{conv}\{u_i\}$, we have $0 \in \operatorname{int}\operatorname{conv}\{u_i\}$, while $u_i, \hat{u}_i \in \operatorname{bd}\operatorname{conv}\{u_i\}$; so the origin lies between u_i and $\hat{u}_i$. For each i, let $\hat{u}_i = -\lambda_i u_i$, where $\lambda_i > 0$. Note that

$$0 = \frac{\lambda_i}{\lambda_i + 1} u_i + \frac{1}{\lambda_i + 1} \hat{u}_i \tag{1}$$

gives the origin as a convex combination of u_i and $\hat{u}_i$. Note also that

$$r \le |\hat{u}_i| = \lambda_i |u_i| \le \lambda_i \qquad \text{for all } i, \tag{2}$$

since $rB_2^n \subseteq \operatorname{conv}\{u_i\} \subseteq B_2^n$.

Now, in the barycentric coordinate system associated with this simplex, the ith coordinate of u_i is 1 and the ith coordinate of $\hat{u}_i$ is 0. Since the barycentric map — the map which assigns, to each point of $\mathbb{R}^n$, the $(n+1)$-tuple containing the barycentric coordinates of that point — is affine, it preserves affine combinations such as (1); therefore the ith coordinate of the origin is $\frac{\lambda_i}{\lambda_i+1}$. Since the sum of a point's barycentric coordinates is 1, we have

$$1 = \sum_{i=1}^{n+1} \frac{\lambda_i}{\lambda_i + 1} \ge \sum_{i=1}^{n+1} \frac{r}{r+1} = \frac{(n+1)r}{r+1} . \tag{3}$$

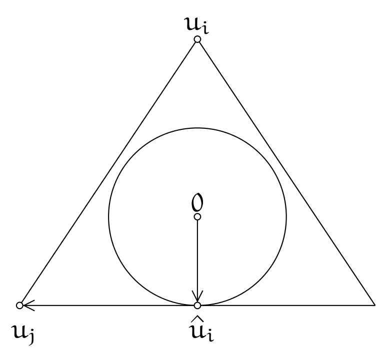
Solving for r yields $r \leq \frac{1}{n}$, as claimed.

We have already seen that the unit simplex contains $\frac{1}{n}B_2^n$; it remains to show that no other simplex does.

So suppose $r = \frac{1}{n}$. Then we have equality in (3), and so $\lambda_i = r$ for all i. Thus we have equality in (2), whence all $|u_i| = 1$, that is,

$$\langle u_i, u_i \rangle = 1 \qquad \text{for all } i. \tag{4}$$

Equality in (2) also yields $|\hat{u}_i| = r$ for all i, so the facet opposite u_i meets rB_2^n at $\hat{u}_i$. Since rB_2^n lies in one of the half-spaces defined by this facet (otherwise $rB_2^n \not\subseteq \operatorname{conv}\{u_i\}$), this facet is a supporting hyperplane of rB_2^n. Now, if $i \neq j$, then the vector from $\hat{u}_i$ to u_j lies in the facet opposite u_i:



Since a supporting hyperplane of a Euclidean ball is orthogonal to the radius to the point of contact,

$$0 = \langle \hat{u}_i, u_j - \hat{u}_i \rangle = \langle -ru_i, u_j + ru_i \rangle = -r\langle u_i, u_j \rangle - r^2 ,$$

whence

$$\langle u_i, u_j \rangle = -r = -\frac{1}{n} \qquad \text{if } i \neq j.$$

In combination with (4), this establishes that the u_i are the vertices of a unit simplex, as desired. □