Identities on intersection and union with a fixed set

For any sets A, B, and S,

$$A = B \iff A \cap S = B \cap S \text{ and } A \cup S = B \cup S\,. \tag{1}$$

The $\Rightarrow$ direction is obvious: $X \cap S$ is a function of X, so it attains equal values for equal values of X; similarly for $X \cup S$.

The $\Leftarrow$ direction is perhaps more surprising. It can certainly occur that $A \neq B$ but $A \cap S = B \cap S$, that is, that A and B differ but we can't tell the difference looking only their intersections with S; in this sense, taking intersection with S loses information. Likewise we can have $A \neq B$ but $A \cup S = B \cup S$, so taking union with S loses information. The $\Leftarrow$ direction tells us that these two operations lose complementary information. The word complement is completely appropriate here: the information lost by taking intersection with S is just how A and B differ outside S; the information lost by taking union with S is just how A and B differ inside S.

This intuitive account of the $\Leftarrow$ direction can be transformed into a proof, which in its most elegant form exploits again the principle that a function attains equal values for equal arguments. Indeed, we can express X as a function of $X \cap S$ and $X \cup S$ as follows:

$$\begin{aligned} X &= X \cap 1 \\ &= X \cap (S \cup \bar{S}) \\ &= (X \cap S) \cup (X \cap \bar{S}) \\ &= (X \cap S) \cup (X \cap \bar{S}) \cup \varnothing \\ &= (X \cap S) \cup (X \cap \bar{S}) \cup (S \cap \bar{S}) \\ &= (X \cap S) \cup ((X \cup S) \cap \bar{S}) \end{aligned}$$

Here 1 denotes the universe, $\varnothing$ denotes the empty set, and $\bar{}$ denotes complement with respect to the universe.

In the same spirit but another vein, we can use the symmetric difference $\triangle$, defined by

$$A \triangle B = (A \cap \bar{B}) \cup (B \cap \bar{A})\,,$$

among whose many properties is the fact that $A \triangle B = \varnothing$ exactly when $A = B$. Thus (1) can be rewritten as

$$A \triangle B = \varnothing \iff (A \cap S) \triangle (B \cap S) = \varnothing \text{ and } (A \cup S) \triangle (B \cup S) = \varnothing\,.$$

This version of (1) follows immediately from the very pleasant identity

$$A \triangle B = ((A \cap S) \triangle (B \cap S)) \cup ((A \cup S) \triangle (B \cup S))\,,$$

proving which is a good exercise in the properties of the symmetric difference.

(Incidentally, (1) is the reason that the complement $\bar{S}$ of a set S is uniquely defined by the conditions that $S \cap \bar{S} = \varnothing$ and $S \cup \bar{S} = 1$.)