See-saw approximations with the gamma function integral

As is well-known, integration by parts and induction on n yield

$$\int_0^\infty t^n e^{-t}\,dt = n!$$

for all natural numbers (i.e., nonnegative integers) n. Applying the same technique to a proper integral of the same type yields

$$\frac{1}{n!}\int_0^x t^n e^{-t}\,dt + \frac{1}{e^x}\sum_{k=0}^{n}\frac{x^k}{k!} = 1$$

for all natural numbers n and all real numbers x.

Note that $\sum_{k=0}^{n}\frac{x^k}{k!}$ is a truncation of the Maclaurin series for e^x; conceive it as an approximation to e^x. The second term on the left-hand side of this equation expresses the relative accuracy of that approximation. In the same way, the first term expresses the relative accuracy of $\int_0^x t^n e^{-t}\,dt$ as an approximation to $n!$. The equation asserts that these approximations are in a see-saw relationship: the one approximation is as good as the other is bad. (This interpretation is satisfying only for $x \geq 0$, when both terms lie in $[0, 1]$.)