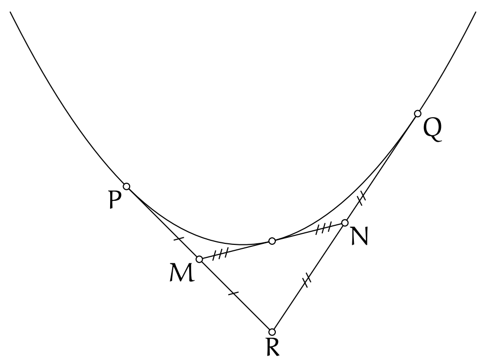
Notes on tangents to parabolas

(These are notes for a talk I gave on 2007 March 30.)

The point of this talk is not to publicize new results. The most recent material in it is the concept of Bézier curves, which dates to the late 1950s; the second most recent material is the quadrature of the parabola, which is a famous result of Archimedes from 250 BC or so. The point of this talk is, rather, to advertise synthetic geometry, that is, geometry without coordinates, as practiced by Euclid.

Consider, for example, the following proposition: Let the tangents to a parabola at P and Q meet at R. Let M and N be the midpoints of PR and RQ respectively. Then MN is also tangent to the parabola, and the point of tangency is the midpoint of MN.



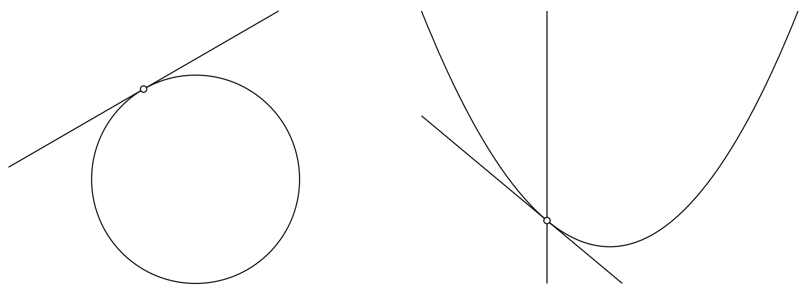
We'll prove this below. For now, just notice how thoroughly routine it is to prove it with modern techniques. Adopting an appropriate coordinate system, the parabola has equation $y = x^2$. Let P be the point (p, p^2), and let Q be the point (q, q^2). Now just compute everything: deploy the techniques of first-year calculus to find the equations of the tangent lines; solve a two-equation linear system to find the coordinates of their intersection; and so on.

Those computations are not totally without interest, but they don't leave me feeling that I understand the phenomenon. A synthetic approach yields, at least, an instructively different view of it.

The first order of business is to understand what a tangent is, a question that turns out to be somewhat thornier than you might expect.

In calculus we have a definition of tangents (of graphs of functions, at first, and later of more general curves) in terms of derivatives, but this requires a large apparatus of theory which involves coordinates at a fundamental level, and is rather in the wrong spirit for what we're doing.

In Euclidean geometry one usually deals with tangents to circles, and for circles we have a simple definition: a line is tangent to a circle if it meets the circle at exactly one point. This definition will not, however, serve for parabolas. We do indeed expect that a tangent to a parabola will meet the parabola only at the point of tangency, but so too will a line parallel to the axis of the parabola.[1]



Euclid's own definition of tangents is slightly more helpful: "A straight line is said to *touch a circle* which, meeting the circle and being produced, does not cut the circle."[2] That is, a tangent is a line that *meets* but does not *cut*.

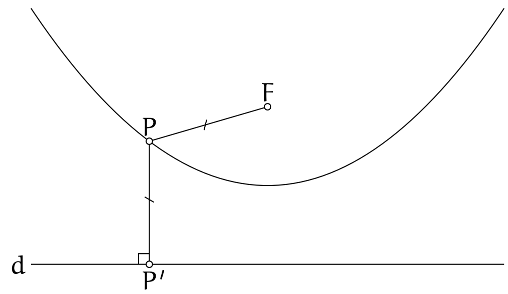
"Meeting" is easy to understand: lines and other curves are sets of points, and they "meet" at a point if they both contain that point. "Cutting" is a bit more mysterious; Euclid doesn't define it. The simplest interpretation I can think of is this: a line divides the plane into two regions; the line, we say, *cuts* a figure if the figure contains points in both of those regions.[3]

There are a few ways to define parabolas; for our purposes, the simplest one is this: We are given a point F, called the focus of the parabola, and a line d, called its directrix. The parabola consists of those points which are equidistant from the focus and the directrix.

[1]The righteous way to deal with this problem is actually found in projective geometry; there, the parabola and a line parallel to its axis *do* meet at a second point, a "point at infinity", so we can define tangents for parabolas as we do for circles. (Indeed, in projective geometry all conics are the same.)

[2]Euclid's *Elements*, Book III, Definition 3, quoted from *The Thirteen Books of Euclid's Elements*, trans. and ed. Thomas L. Heath, 2nd ed. (New York: Dover, 1956), 2:1.

[3]This definition of "cut" means that a line that meets but does not cut is, in the terminology of convex geometry, a "supporting hyperplane", not a tangent. We can get away with this because the interior of the parabola is convex. (Proving that is a good exercise.)



As usual, when we speak of distance between a point and a line, we mean the perpendicular distance. Thus, to determine whether a point P is on the parabola, we drop the perpendicular from P to d, meeting d at P'; then we compare the distance PF to the distance PP'. P is on the parabola if and only if $PF = PP'$.

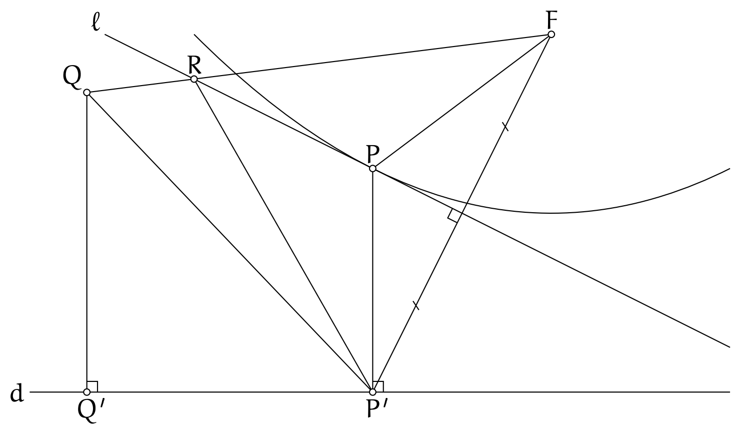
We will be dropping perpendiculars to d quite often in what follows. The foot of the perpendicular will always be named with the prime symbol; that is, for any point X, the foot of the perpendicular dropped from X to d will be called X'.

Now, the fact that a point P on the parabola is equidistant from P' and F has a familiar consequence: P lies on the perpendicular bisector of the segment $P'F$. Our first theorem asserts that this line not only passes through P but is the tangent to the parabola there.

One-Tangent Theorem Let P lie on the parabola. The perpendicular bisector of $P'F$ is the tangent to the parabola at P.

Proof Let ℓ be the perpendicular bisector of $P'F$. We have already seen that ℓ passes through P, so ℓ meets the parabola. It remains to show that ℓ does not cut the parabola.

We will show that every point on the parabola is on the same side of ℓ as F; equivalently, that no point on the opposite side of ℓ from F is on the parabola.



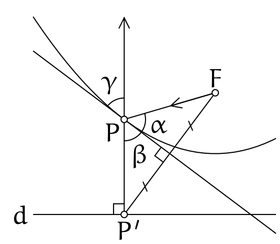
Let Q lie on the opposite side of ℓ from F. Join QF, meeting ℓ at R. We have

$$\begin{aligned} QF &= QR + RF && \text{(R is between Q and F)} \\ &= QR + RP' && \text{(R is on } \ell\text{)} \\ &> QP' && \text{(triangle inequality; see below)} \\ &\geq QQ' && \text{(perpendicular is shortest)} \end{aligned}$$

and so Q does not lie on the parabola.

The inequality $QR + RP' > QP'$ is strict because R does not lie on the segment QP'. For since ℓ is, by construction, the perpendicular bisector of $P'F$, P' and F lie on opposite sides of ℓ. Therefore Q and P' lie on the same side of ℓ, that is, the segment QP' does not intersect ℓ. In particular, the segment QP' does not contain R. □

This proof is, in fact, not complete. It does show that, by our definitions, the line described is tangent to the parabola at P; but the theorem asserts that it is *the* tangent to the parabola at P. We have not seen any argument that the parabola has only one tangent at each point. The only proof I have of this statement (in the style of geometry we're doing) is quite long and involved, so I omit it. If you know of a simple proof, I'd like to hear about it.

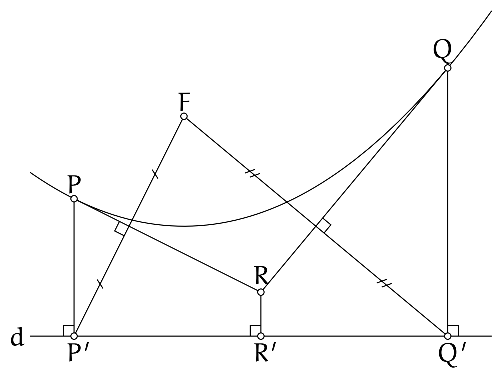


As an application of the One-Tangent Theorem, we can prove the famous optical property of the parabola. In the figure at right, $\alpha = \beta$ since the tangent bisects $\angle FPP'$, and $\beta = \gamma$ as vertical angles. So $\alpha = \gamma$, which since light bounces with the angle of incidence equal to the angle of reflection entails that a beam of light emanating from F and hitting the parabola at P will bounce off along the (extension of the) line PP'. And, of course, in reverse: light arriving perpendicular to the directrix will be reflected to the focus.

The following theorem gives another example of how the One-Tangent Theorem turns knowledge of perpendicular bisectors into knowledge of tangents to parabolas.

Two-Tangent Theorem Let P and Q lie on the parabola, and let the tangents at P and Q meet at R. Then R' bisects $P'Q'$.

Proof By the One-Tangent Theorem, PR is the perpendicular bisector of $P'F$, and QR is the perpendicular bisector of $Q'F$. Therefore their intersection R is the circumcentre of $\triangle P'FQ'$, and RR', being the perpendicular from R to $P'Q'$, is also the perpendicular bisector of $P'Q'$.

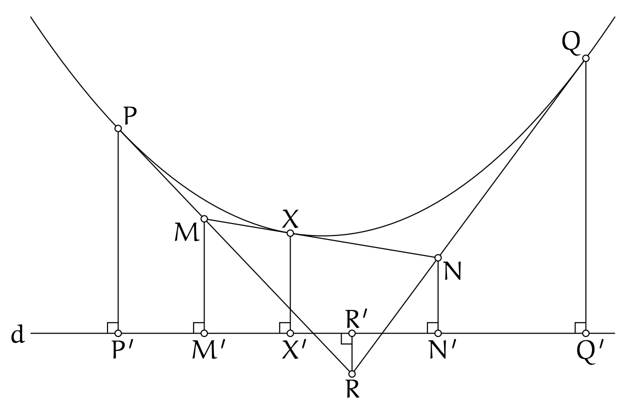


□

Three-Tangent Theorem Let P, Q, and X lie on the parabola. Let the tangents at P and Q meet at R. Let the tangent at X meet PR at M and RQ at N. Then

$$\frac{PM}{MR} = \frac{MX}{XN} = \frac{RN}{NQ} .$$

Proof



By the Two-Tangent Theorem (for the tangents PM and MX), $P'M' = M'X'$. Furthermore,

$$\begin{aligned}
P'M' &= \tfrac{1}{2}P'X' && \text{(Two-Tangent Th.: } PM, MX\text{)} \\
&= \tfrac{1}{2}P'Q' - \tfrac{1}{2}X'Q' \\
&= R'Q' - N'Q' && \text{(Two-Tangent Th.: } PR, RQ\text{; and } XN, NQ\text{)} \\
&= R'N' .
\end{aligned}$$

Thus $P'M' = M'X' = R'N'$. Similarly, $M'R' = X'N' = N'Q'$, so

$$\frac{P'M'}{M'R'} = \frac{M'X'}{X'N'} = \frac{R'N'}{N'Q'} .$$

The desired equalities now follow by parallels. □

Letting $t = \frac{MR}{PR}$ (so that the proportion mentioned in the statement of the Three-Tangent Theorem is $(1+t)^{-1}$) and considering the points as vectors (so we can apply vector arithmetic), we have

$$\begin{aligned}
M &= (1-t)P + tR , \\
N &= (1-t)R + tQ , \\
\text{and} \qquad X &= (1-t)M + tN ,
\end{aligned}$$

whence

$$X = (1-t)^2 P + 2t(1-t)R + t^2 Q .$$

As t varies from 0 to 1, the point X varies along the parabola from P to Q. Notice that the coefficients of this linear combination of P, Q, and R arise from the expansion of the square of a binomial:

$$((1-t)+t)^2 = (1-t)^2 + 2t(1-t) + t^2 .$$

It is, then, natural to consider analogous parameterizations of curves based on the expansion of other powers of a binomial. For example, considering the cube of a binomial leads to the curve

$$(1-t)^3 u + 3t(1-t)^2 v + 3t^2(1-t)w + t^3 x$$

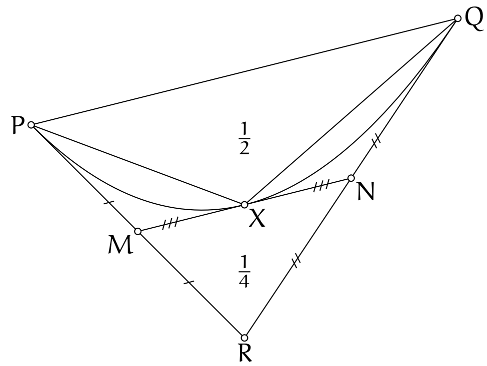
(where u, v, w, and x are fixed vectors). The families of curves obtained by such parameterizations are known as Bézier curves, and are much used in computer graphics and computer-aided design.[4]

The Three-Tangent Theorem can also be applied to prove a famous result of Archimedes.[5] Taking the special case where the proportion in the theorem

[4]A great deal of information about these curves can be found by web search.

[5]See his *Quadrature of the Parabola*, in *The Works of Archimedes*, trans. and ed. Thomas L. Heath (New York: Dover, 2002), 233–52. He gives two proofs; the one in the text differs from both. Also, his statement of the result is that the area of the parabolic segment is $\frac{4}{3}$ the area of $\triangle PQX$.

is 1 yields the corollary mentioned at the beginning of these notes: Let the tangents to a parabola at P and Q meet at R. Let M and N be the midpoints of PR and RQ respectively. Then MN is also tangent to the parabola, and the point of tangency X is the midpoint of MN.



Let the area of $\triangle PQR$, the triangle formed by the two tangents and the secant, be 1. The parabolic "segment" PQX includes the triangle $\triangle PQX$, with area $\frac{1}{2}$, and excludes the triangle $\triangle MNR$, with area $\frac{1}{4}$. The remaining pieces, $\triangle PMX$ and $\triangle XNQ$, are two smaller instances of the same situation: they are triangles formed by two tangents and a secant. By the analogous subdivision of these triangles, the parabolic segment includes $\frac{1}{2}$ of their area and excludes $\frac{1}{4}$ of it, leaving four new smaller instances of the same situation again. Continuing in this manner, we see that the area of the parabolic segment is $\frac{1}{2}/(\frac{1}{2}+\frac{1}{4}) = \frac{2}{3}$ of the area of $\triangle PQR$.