Measurable sets with non-measurable Minkowski sum

Some of the results of [2], somewhat reorganized. Throughout, \( \mathbb{R} \) is endowed with Lebesgue measure and considered as a vector space over \( \mathbb{Q} \) (so that when we speak of linear combinations we mean linear combinations with rational coefficients, and likewise for the notions of linear dependence, span, and basis).

**Proposition 1** Any set of positive measure contains two distinct points whose difference is rational.

**Proof** Let \( A \subseteq \mathbb{R} \) have positive measure. Then there is some bounded interval \( I \) such that \( A \cap I \) has positive measure. If \( A \) had no points such as desired, then the sets \( (A \cap I + \frac{1}{n} : n \in \mathbb{N}) \) would be pairwise disjoint. But then the union of these sets would have infinite measure, despite being contained in an interval. \( \square \)

**Remark 2** Steinhaus’s theorem asserts that the difference set of a set of positive measure contains an interval around the origin, which immediately implies the proposition; but Sierpiński proves the statement as above. For references and a simple proof of Steinhaus’s theorem, see [3]. (Steinhaus’s original paper on this subject is actually in the same issue of Fundamenta Mathematicae as, and immediately precedes, Sierpiński’s paper [2].)

**Corollary 3** Any set of positive measure spans \( \mathbb{R} \).

**Proof** Let \( A \subseteq \mathbb{R} \) have positive measure. Let \( x \in \mathbb{R} \). If \( x = 0 \) then \( x \in \text{span} A \); assume \( x \neq 0 \). Then \( \frac{1}{x} A \) has positive measure; let \( a, b \in A \) be distinct and such that \( \frac{a}{x} - \frac{b}{x} = q \in \mathbb{Q} \). Since \( a \neq b \), we have \( q \neq 0 \), and so \( x = (a - b)/q \in \text{span} A \). \( \square \)

**Remark 4** As Sierpiński notes, if \( \mathbb{R} \) has a basis, then it follows from the above result that nonmeasurable sets exist. Indeed, if \( B \) is a basis, choose \( b \in B \) and let \( V = \text{span} (B \setminus \{ b \}) \). Then \( V \) cannot have positive measure because it is a proper subspace of \( \mathbb{R} \), and it cannot have zero measure because \( \mathbb{R} = \bigcup_{q \in \mathbb{Q}} (V + q b) \) is a countable union of translates of \( V \).

**Corollary 5** Any measurable basis for \( \mathbb{R} \) has measure zero.

**Proof** By contraposition. Suppose \( A \subseteq \mathbb{R} \) has positive measure. Let \( a \in A \). Then \( A \setminus \{ a \} \) has positive measure, and so \( a \in \mathbb{R} = \text{span}(A \setminus \{ a \}) \). Therefore \( A \) is linearly dependent. \( \square \)

**Proposition 6** \( \mathbb{R} \) has a measurable basis.

**Proof** Let \( X \) be the set of real numbers whose binary expansions have zeroes in even-numbered places after the binary point; let \( Y \) be the set of real num-

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bers whose binary expansions have zeroes in odd-numbered places after the binary point. Let \( B \) be a subset of \( X \cup Y \) which is linearly independent and maximal for this condition. (I use Zorn’s lemma here; Sierpiński gives a slightly different argument at this stage, using Zermelo’s theorem that all sets can be well-ordered.) By maximality \( B \) spans \( X \cup Y \), hence spans \( \mathbb{R} \); it is linearly independent by construction; it has measure zero because \( X \) and \( Y \) do, and so in particular is measurable. \( \square \)

**Remark 7** Sierpiński cites [1] for the statement that there exists a basis which meets every perfect set. Such a basis cannot have measure zero, since every set of full measure contains a perfect set; therefore such a basis is nonmeasurable.

**Proposition 8** There exist measurable sets \( X, Y \subseteq \mathbb{R} \) such that \( X + Y \) is not measurable.

**Proof** Let \( B \) be a measurable basis for \( \mathbb{R} \). Fix \( b \in B \) and let \( V = \text{span}(B \setminus \{b\}) \). For each \( n \in \mathbb{N} \cup \{0\} \), let \( A_n \) be the set of real numbers with at most \( n \) nonzero coordinates in the basis \( B \). (Note that \( A_0 = \{0\} \).)

Now, assume for contradiction that sums of measurable sets are measurable. Since \( A_{n+1} = A_n + \bigcup_{q \in \mathbb{Q}} qB \), it follows by induction that all \( A_n \) are measurable. Since \( \mathbb{R} = \bigcup_{n \in \mathbb{N}} A_n \), some \( A_n \) has positive measure; let \( k \in \mathbb{N} \) be the least natural number such that \( A_k \) has positive measure.

Now, if \( q \in \mathbb{Q} \) and \( q \neq 0 \), then \( A_k \cap (V + qb) = A_{k-1} \cap V + qb \subseteq A_{k-1} + qb \), which is a translate of \( A_{k-1} \) and so has zero measure. Therefore \( A_k \cap (V + qb) \) has zero measure. But then \( A_k \cap V = A_k \setminus \bigcup_{q \neq 0} (A_k \cap (V + qb)) \) has the same measure as \( A_k \), in particular, positive measure, and so \( A_k \cap V \) spans \( \mathbb{R} \). But this is absurd, since \( V \) is a proper subspace of \( \mathbb{R} \). \( \square \)

**References**

