Measurable sets with non-measurable Minkowski sum

Some of the results of [2], somewhat reorganized. Throughout, $\mathbb{R}$ is endowed with Lebesgue measure and considered as a vector space over $\mathbb{Q}$ (so that when we speak of linear combinations we mean linear combinations with rational coefficients, and likewise for the notions of linear dependence, span, and basis).

Proposition 1 Any set of positive measure contains two distinct points whose difference is rational.

Proof Let $A \subseteq \mathbb{R}$ have positive measure. Then there is some bounded interval I such that $A \cap I$ has positive measure. If A had no points such as desired, then the sets $(A \cap I + \frac{1}{n} : n \in \mathbb{N})$ would be pairwise disjoint. But then the union of these sets would have infinite measure, despite being contained in an interval. □

Remark 2 Steinhaus's theorem asserts that the difference set of a set of positive measure contains an interval around the origin, which immediately implies the proposition; but Sierpiński proves the statement as above. For references and a simple proof of Steinhaus's theorem, see [3]. (Steinhaus's original paper on this subject is actually in the same issue of Fundamenta Mathematicae as, and immediately precedes, Sierpiński's paper [2].)

Corollary 3 Any set of positive measure spans $\mathbb{R}$.

Proof Let $A \subseteq \mathbb{R}$ have positive measure. Let $x \in \mathbb{R}$. If $x = 0$ then $x \in \operatorname{span} A$; assume $x \neq 0$. Then $\frac{1}{x}A$ has positive measure; let $a, b \in A$ be distinct and such that $\frac{a}{x} - \frac{b}{x} = q \in \mathbb{Q}$. Since $a \neq b$, we have $q \neq 0$, and so $x = (a-b)/q \in \operatorname{span} A$. □

Remark 4 As Sierpiński notes, if $\mathbb{R}$ has a basis, then it follows from the above result that nonmeasurable sets exist. Indeed, if B is a basis, choose $b \in B$ and let $V = \operatorname{span}(B \setminus \{b\})$. Then V cannot have positive measure because it is a proper subspace of $\mathbb{R}$, and it cannot have zero measure because $\mathbb{R} = \bigcup_{q \in \mathbb{Q}}(V + qb)$ is a countable union of translates of V.

Corollary 5 Any measurable basis for $\mathbb{R}$ has measure zero.

Proof By contraposition. Suppose $A \subseteq \mathbb{R}$ has positive measure. Let $a \in A$. Then $A \setminus \{a\}$ has positive measure, and so $a \in \mathbb{R} = \operatorname{span}(A \setminus \{a\})$. Therefore A is linearly dependent. □

Proposition 6 $\mathbb{R}$ has a measurable basis.

Proof Let X be the set of real numbers whose binary expansions have zeroes in even-numbered places after the binary point; let Y be the set of real numbers whose binary expansions have zeroes in odd-numbered places after the binary point. Let B be a subset of $X \cup Y$ which is linearly independent and maximal for this condition. (I use Zorn's lemma here; Sierpiński gives a slightly different argument at this stage, using Zermelo's theorem that all sets can be well-ordered.) By maximality B spans $X \cup Y$, hence spans $\mathbb{R}$; it is linearly independent by construction; it has measure zero because X and Y do, and so in particular is measurable. $\square$

Remark 7 Sierpiński cites [1] for the statement that there exists a basis which meets every perfect set. Such a basis cannot have measure zero, since every set of full measure contains a perfect set; therefore such a basis is nonmeasurable.

Proposition 8 There exist measurable sets $X, Y \subseteq \mathbb{R}$ such that $X + Y$ is not measurable.

Proof Let B be a measurable basis for $\mathbb{R}$. Fix $b \in B$ and let $V = \operatorname{span}(B \setminus \{b\})$. For each $n \in \mathbb{N} \cup \{0\}$, let A_n be the set of real numbers with at most n nonzero coordinates in the basis B. (Note that $A_0 = \{0\}$.)

Now, assume for contradiction that sums of measurable sets are measurable. Since $A_{n+1} = A_n + \bigcup_{q \in \mathbb{Q}} qB$, it follows by induction that all A_n are measurable. Since $\mathbb{R} = \bigcup_{n \in \mathbb{N}} A_n$, some A_n has positive measure; let $k \in \mathbb{N}$ be the least natural number such that A_k has positive measure.

Now, if $q \in \mathbb{Q}$ and $q \neq 0$, then $A_k \cap (V + qb) = A_{k-1} \cap V + qb \subseteq A_{k-1} + qb$, which is a translate of A_{k-1} and so has zero measure. Therefore $A_k \cap (V + qb)$ has zero measure. But then $A_k \cap V = A_k \setminus \bigcup_{q \neq 0} (A_k \cap (V + qb))$ has the same measure as A_k, in particular, positive measure, and so $A_k \cap V$ spans $\mathbb{R}$. But this is absurd, since V is a proper subspace of $\mathbb{R}$. $\square$

References

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