Euler's formula without calculus

Steven Taschuk

Intersections $K \cap W$

2014 September 16

Euler's formula

$$e^{iv} = \cos v + i \sin v \qquad (i^2 = -1)$$

$$\frac{\left(1+\frac{v\sqrt{-1}}{i}\right)^i + \left(1-\frac{v\sqrt{-1}}{i}\right)^i}{2}$$; atque $\mathit{ſin.}\, v =$

$$\frac{\left(1+\frac{v\sqrt{-1}}{i}\right)^i - \left(1-\frac{v\sqrt{-1}}{i}\right)^i}{2\sqrt{-1}}.$$ In Capite autem præcedente vidimus eſſe $\left(1+\frac{z}{i}\right)^i = e^z$, denotante e baſin Logarithmorum hyperbolicorum: ſcripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit $\mathit{coſ.}\, v =$

$$\frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2} \;\&\; \mathit{ſin.}\, v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}.$$

Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Coſinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} = \mathit{coſ.}\, v + \sqrt{-1}.\,\mathit{ſin.}\, v$ & $e^{-v\sqrt{-1}} = \mathit{coſ.}\, v - \sqrt{-1}.\,\mathit{ſin.}\, v$.

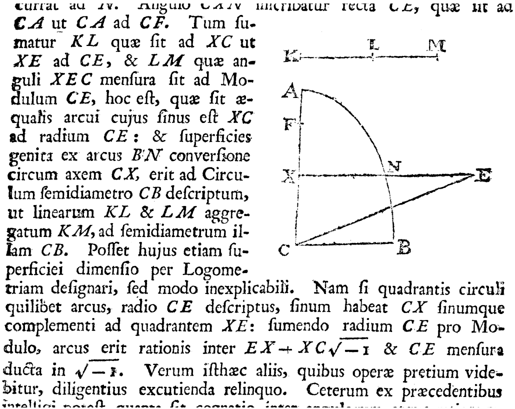
139. Sit jam in iiſdem formulis §. 130. n numerus infinite parvus, ſeu $n = \frac{1}{i}$, exiſtente i numero infinite magno, erit

Introductio in Analysin Infinitorum (1748), p. 104

Leonhard Euler
(1707–1783)

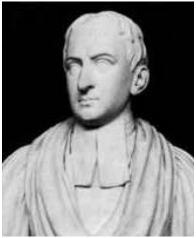
Cotes–Euler formula

$$e^{iv} = \cos v + i \sin v \qquad (i^2 = -1)$$



currat ad N. Angulo CXN inſcribatur recta CE, quæ ſit ad CA ut CA ad CF. Tum ſumatur KL quæ ſit ad XC ut XE ad CE, & LM quæ anguli XEC menſura ſit ad Modulum CE, hoc eſt, quæ ſit æqualis arcui cujus ſinus eſt XC ad radium CE: & ſuperficies genita ex arcus BN converſione circum axem CX, erit ad Circulum ſemidiametro CB deſcriptum, ut linearum KL & LM aggregatum KM, ad ſemidiametrum illam CB. Poſſet hujus etiam ſuperficiei dimenſio per Logometriam deſignari, ſed modo inexplicabili. Nam ſi quadrantis circuli quilibet arcus, radio CE deſcriptus, ſinum habeat CX ſinumque complementi ad quadrantem XE: ſumendo radium CE pro Modulo, arcus erit rationis inter $EX + XC\sqrt{-1}$ & CE menſura ducta in $\sqrt{-1}$. Verum iſthæc aliis, quibus operæ pretium videbitur, diligentius excutienda relinquo. Ceterum ex præcedentibus intelligi poteſt ...

"Logometria" (1714), p. 31



Roger Cotes (1682–1716)

Cotes–Euler formula: a proof

For any real number x,

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Assume this holds for complex exponents too. Then

$$\begin{aligned} e^{iv} &= \frac{(iv)^0}{0!} + \frac{(iv)^1}{1!} + \frac{(iv)^2}{2!} + \frac{(iv)^3}{3!} + \frac{(iv)^4}{4!} + \frac{(iv)^5}{5!} + \cdots \\ &= \frac{v^0}{0!} + \frac{iv^1}{1!} - \frac{v^2}{2!} - \frac{iv^3}{3!} + \frac{v^4}{4!} + \frac{iv^5}{5!} - \frac{v^6}{6!} - \frac{iv^7}{7!} + \cdots \\ &= \cos v + i \sin v \end{aligned}$$

Elementary definition of exponentiation

$$a^n = \overbrace{a \cdot a \cdots a}^{n \text{ copies}}$$

Elementary definition of exponentiation

$$a^n = \overbrace{a \cdot a \cdots a}^{n \text{ copies}}$$

$$a^0 = 1$$

Elementary definition of exponentiation

$$a^n = \overbrace{a \cdot a \cdots a}^{n \text{ copies}}$$

$$a^0 = 1$$

$$a^{-n} = \frac{1}{a^n}$$

Elementary definition of exponentiation

$$
\begin{aligned}
a^n &= \overbrace{a \cdot a \cdots a}^{n \text{ copies}} \\
a^0 &= 1 \\
a^{-n} &= \frac{1}{a^n} \\
a^{m/n} &= \sqrt[n]{a^m}
\end{aligned}
$$

Elementary definition of exponentiation

$$
\begin{aligned}
a^n &= \overbrace{a \cdot a \cdots a}^{n \text{ copies}} \\
a^0 &= 1 \\
a^{-n} &= \frac{1}{a^n} \\
a^{m/n} &= \sqrt[n]{a^m} \\
a^x &= \lim_{\frac{m}{n} \to x} a^{m/n}
\end{aligned}
$$

Reason for defining a^0 and a^{-n} as we do

$$\begin{aligned}
a^4 &= a \cdot a \cdot a \cdot a \\
a^3 &= a \cdot a \cdot a && \uparrow \times a \\
a^2 &= a \cdot a && \downarrow \div a \\
a^1 &= a
\end{aligned}$$

Reason for defining a^0 and a^{-n} as we do

$$a^4 = a \cdot a \cdot a \cdot a$$

$$a^3 = a \cdot a \cdot a \qquad \uparrow \times a$$

$$a^2 = a \cdot a \qquad \downarrow \div a$$

$$a^1 = a$$

$$a^0 = 1$$

Reason for defining a^0 and a^{-n} as we do

$$
\begin{aligned}
a^4 &= a \cdot a \cdot a \cdot a \\
a^3 &= a \cdot a \cdot a & \uparrow \times a \\
a^2 &= a \cdot a & \downarrow \div a \\
a^1 &= a \\
a^0 &= 1 \\
a^{-1} &= \frac{1}{a}
\end{aligned}
$$

Reason for defining a^0 and a^{-n} as we do

$$a^4 = a \cdot a \cdot a \cdot a$$

$$a^3 = a \cdot a \cdot a \qquad \uparrow \times a$$

$$a^2 = a \cdot a \qquad \downarrow \div a$$

$$a^1 = a$$

$$a^0 = 1$$

$$a^{-1} = \frac{1}{a}$$

$$a^{-2} = \frac{1}{a^2}$$

Reason for defining a^0 and a^{-n} as we do

$$\begin{aligned} a^4 &= a \cdot a \cdot a \cdot a \\ a^3 &= a \cdot a \cdot a && \uparrow \times a \\ a^2 &= a \cdot a && \downarrow \div a \\ a^1 &= a \\ a^0 &= 1 \\ a^{-1} &= \frac{1}{a} \\ a^{-2} &= \frac{1}{a^2} \end{aligned}$$

i.e., we assume $a^{n+1} = a^n \cdot a$ is also valid for negative exponents.

Functional def'n of exponentiation (integer exponents)

Theorem

For every positive number a,
there is exactly one function $f : \mathbb{Z} \to \mathbb{R}$
such that

- $f(1) = a$, *and*
- $f(m+n) = f(m)f(n)$ *for all* $m, n \in \mathbb{Z}$.

We write $f(n) = a^n$.

Functional def'n of exponentiation (rational exponents)

Theorem

For every positive number a,
there is exactly one function $f : \mathbb{Q} \to \mathbb{R}$
such that

- $f(1) = a$, *and*
- $f(p+q) = f(p)f(q)$ *for all* $p, q \in \mathbb{Q}$.

We write $f(q) = a^q$.

Functional def'n of exponentiation (real exponents)

Theorem

For every positive number a,
there is exactly one function $f : \mathbb{R} \to \mathbb{R}$
such that

- $f(1) = a$, *and*
- $f(u + v) = f(u)f(v)$ *for all* $u, v \in \mathbb{R}$.

We write $f(v) = a^v$.

FALSE (assuming the axiom of choice)

Functional def'n of exponentiation (real exponents)

Theorem

For every positive number a,
there is exactly one function $f : \mathbb{R} \to \mathbb{R}$
such that

- $f(1) = a$, *and*
- $f(u + v) = f(u)f(v)$ *for all* $u, v \in \mathbb{R}$, *and*
- f *is continuous.*

We write $f(v) = a^v$.

TRUE

Cotes–Euler formula: a proof

For any real number x,

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Assume this holds for complex exponents too. Then

$$\begin{aligned} e^{iv} &= \frac{(iv)^0}{0!} + \frac{(iv)^1}{1!} + \frac{(iv)^2}{2!} + \frac{(iv)^3}{3!} + \frac{(iv)^4}{4!} + \frac{(iv)^5}{5!} + \cdots \\ &= \frac{v^0}{0!} + \frac{iv^1}{1!} - \frac{v^2}{2!} - \frac{iv^3}{3!} + \frac{v^4}{4!} + \frac{iv^5}{5!} - \frac{v^6}{6!} - \frac{iv^7}{7!} + \cdots \\ &= \cos v + i \sin v \end{aligned}$$

Cotes–Euler formula without calculus: part 0

$$e^{u+iv} = e^u e^{iv}$$

Cotes–Euler formula without calculus: part 1

Let $e^{iv} = C(v) + iS(v)$; want to show $C = \cos$, $S = \sin$.

$$e^{i(u+v)} = e^{iu+iv} = e^{iu}e^{iv}$$

$$\begin{aligned} C(u+v) + iS(u+v) &= (C(u) + iS(u))(C(v) + iS(v)) \\ &= (C(u)C(v) - S(u)S(v)) \\ &\qquad + i(S(u)C(v) + C(u)S(v)) \end{aligned}$$

$$\begin{aligned} C(u+v) &= C(u)C(v) - S(u)S(v) \\ S(u+v) &= S(u)C(v) + C(u)S(v) \end{aligned}$$

Cotes–Euler formula without calculus: part 2

Theorem

If $C, S\colon \mathbb{R} \to \mathbb{R}$ satisfy

- $C(u+v) = C(u)C(v) - S(u)S(v)$ *for all* $u, v \in \mathbb{R}$, *and*
- $S(u+v) = S(u)C(v) + C(u)S(v)$ *for all* $u, v \in \mathbb{R}$, *and*
- C *and* S *are not both identically zero, and*
- C *and* S *are continuous,*

then there exist $b, \lambda \in \mathbb{R}$, $b > 0$, *such that*

$$C(v) = b^v \cos(\lambda v) \qquad \text{and} \qquad S(v) = b^v \sin(\lambda v)\ .$$

Characterization of sine and cosine: proof outline

1. Isolate the exponential part.
   (Let $f(v) = \sqrt{C(v)^2 + S(v)^2}$; show $f(u+v) = f(u)f(v)$.
   Replace C, S with $C/f, S/f$.)
2. Prove identities, esp. $C(2v) = 2C(v)^2 - 1$.
3. Prove that C has a root (or is constant).
   (If $0 \leq C(v) \leq 1$ then $1 - C(2v) \geq 2(1 - C(v))$;
   so if $C(v) \neq 1$ then some $C(2^n v) < 0$.)
4. Let $\lambda = \frac{\pi}{2p}$, where p is the smallest positive root of C;

successively	extend	$C(v) = \cos(\lambda v)$
	from	$v = p$
	to	v of form $p/2^n$
	to	v of form $mp/2^n$
	to	$v \in \mathbb{R}$

Functional def'n of exponentiation (real exponents)

Theorem

For every positive number a,
there is exactly one function $f : \mathbb{R} \to \mathbb{R}$
such that

- $f(1) = a$, *and*
- $f(u + v) = f(u)f(v)$ *for all* $u, v \in \mathbb{R}$, *and*
- f *is continuous.*

We write $f(v) = a^v$.

Cotes–Euler formula without calculus: part 0

$$e^{u+iv} = e^u e^{iv}$$

Cotes–Euler formula without calculus: part 1

Let $e^{iv} = C(v) + iS(v)$; want to show $C = \cos$, $S = \sin$.

$$e^{i(u+v)} = e^{iu+iv} = e^{iu}e^{iv}$$

$$\begin{aligned} C(u+v) + iS(u+v) &= (C(u) + iS(u))(C(v) + iS(v)) \\ &= (C(u)C(v) - S(u)S(v)) \\ &\qquad + i(S(u)C(v) + C(u)S(v)) \end{aligned}$$

$$\begin{aligned} C(u+v) &= C(u)C(v) - S(u)S(v) \\ S(u+v) &= S(u)C(v) + C(u)S(v) \end{aligned}$$

Cotes–Euler formula without calculus: part 2

Theorem

If $C, S\colon \mathbb{R} \to \mathbb{R}$ satisfy

- $C(u+v) = C(u)C(v) - S(u)S(v)$ *for all* $u, v \in \mathbb{R}$, *and*
- $S(u+v) = S(u)C(v) + C(u)S(v)$ *for all* $u, v \in \mathbb{R}$, *and*
- C *and* S *are not both identically zero, and*
- C *and* S *are continuous,*

then there exist $b, \lambda \in \mathbb{R}$, $b > 0$, *such that*

$$C(v) = b^v \cos(\lambda v) \qquad \text{and} \qquad S(v) = b^v \sin(\lambda v) \ .$$

(Almost) Cotes-Euler formula without calculus

$$e^{iv} = b^v(\cos(\lambda v) + i\sin(\lambda v))$$

(Almost) Cotes-Euler formula without calculus

$$e^{iv} = b^v(\cos(\lambda v) + i\sin(\lambda v))$$

INSERT PICTURES HERE

(Almost) Cotes-Euler formula without calculus

$$e^{iv} = b^v(\cos(\lambda v) + i\sin(\lambda v))$$

INSERT PICTURES HERE

- Getting $b = 1$:
  - further assume $e^{\overline{z}} = \overline{e^z}$

(Almost) Cotes-Euler formula without calculus

$$e^{iv} = b^v(\cos(\lambda v) + i\sin(\lambda v))$$

INSERT PICTURES HERE

- Getting $b = 1$:
  - further assume $e^{\overline{z}} = \overline{e^z}$
- Getting $\lambda = 1$:
  - further assume $(ab)^z = a^z b^z$
  - ...and normalize a logarithm