Cancelling primes in binomial coefficients

Let p be a prime. Then

$$\binom{p}{k} \equiv \begin{cases} 1 \pmod{p} & \text{if } k \in \{0, p\}, \\ 0 \pmod{p} & \text{otherwise.} \end{cases}$$

The first half is trivial: $\binom{p}{0}$ and $\binom{p}{p}$ are equal to 1, so they are certainly congruent to 1 modulo p. Part of the second half is likewise trivial: if $k < 0$ or $k > p$, then $\binom{p}{k} = 0$, which is certainly congruent to 0 modulo p. That leaves the case $0 < k < p$; in this case, Pascal's theorem applies, that is,

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}\,,$$

and so

$$p! = \binom{p}{k} k!(p-k)!\,.$$

The left-hand side includes a factor of the prime p, so by Euclid's lemma p divides one of the factors on the right-hand side. But $k < p$, so all the integers $1, 2, \ldots, k$ that appear in $k!$ are positive and less than p, hence not divisible by p; similarly, since $k > 0$, we have $p - k < p$ and none of the integers that appear in $(p-k)!$ are divisible by p. So p divides $\binom{p}{k}$, that is, $\binom{p}{k} \equiv 0 \pmod{p}$.

Thus if we have n integers $a_1, a_2, \ldots, a_n$, then

$$\binom{p}{a_1}\binom{p}{a_2}\cdots\binom{p}{a_n} \equiv \begin{cases} 1 \pmod{p} & \text{if all } a_i \in \{0, p\}, \\ 0 \pmod{p} & \text{otherwise.} \end{cases}$$

Interpret this product combinatorially: we have n baseball teams, each consisting of p players. We wish to choose a_1 players from team 1, a_2 players from team 2, and so forth; the product gives the number of ways we can do this for a fixed choice of numbers $a_1, a_2, \ldots, a_n$. Note that this procedure chooses $a_1 + a_2 + \cdots + a_n$ players from the total pn players.

Now, if we wish to pick k players altogether, we could just choose them directly, in $\binom{pn}{k}$ ways. Alternatively, we could first decide how many players to take from each team, that is, select numbers $a_1, a_2, \ldots, a_n$ such that $0 \le a_i \le p$ for each i and $a_1 + a_2 + \cdots + a_n = k$, and then choose the players as above. Therefore

$$\binom{pn}{k} = \sum_{\substack{0 \le a_i \le p \\ a_1 + \cdots + a_n = k}} \binom{p}{a_1}\binom{p}{a_2}\cdots\binom{p}{a_n}$$

(where the sum is over all choices for the a_i that satisfy the stated conditions). But as noted above, each term of this sum is congruent modulo p either to 1 (if all its a_i are either 0 or p) or to 0 (otherwise). Thus

$$\binom{pn}{k} \equiv \sum_{\substack{a_i \in \{0,p\} \\ a_1+\cdots+a_n=k}} 1 \pmod{p} .$$

This new sum is easy to evaluate. It counts the number of ways to choose k players from n teams of p players each, where from each team, we take either nobody or everybody. Since we're always taking p players at a time, the total number of players chosen is a multiple of p; that is, when k is not a multiple of p, there are zero ways to get k players under the stated conditions. Thus

$$\binom{pn}{k} \equiv 0 \pmod{p} \quad \text{if } p \nmid k.$$

But if p does divide k, say, $k = pj$, then we can (and must) choose j of the n teams, taking everybody from those teams and nobody from the other teams. Thus

$$\binom{pn}{pj} \equiv \binom{n}{j} \pmod{p} .$$

These congruences express a kind of self-similarity in Pascal's triangle. For example, the cells in rows/columns $0, 5, 10, \ldots$ recapitulate the structure of the whole triangle, modulo 5:

$$\begin{array}{cccccccccccccccc}
\enclose{circle}{1} \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 1 & 4 & 1 \\
\enclose{circle}{1} & 0 & 0 & 0 & 0 & \enclose{circle}{1} \\
1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 2 & 1 & 0 & 0 & 1 & 2 & 1 \\
1 & 3 & 3 & 1 & 0 & 1 & 3 & 3 & 1 \\
1 & 4 & 1 & 4 & 1 & 1 & 4 & 1 & 4 & 1 \\
\enclose{circle}{1} & 0 & 0 & 0 & 0 & \enclose{circle}{2} & 0 & 0 & 0 & 0 & \enclose{circle}{1} \\
1 & 1 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 1 \\
1 & 2 & 1 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 1 & 2 & 1 \\
1 & 3 & 3 & 1 & 0 & 2 & 1 & 1 & 2 & 0 & 1 & 3 & 3 & 1 \\
1 & 4 & 1 & 4 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 4 & 1 & 4 & 1 \\
\enclose{circle}{1} & 0 & 0 & 0 & 0 & \enclose{circle}{3} & 0 & 0 & 0 & 0 & \enclose{circle}{3} & 0 & 0 & 0 & 0 & \enclose{circle}{1}
\end{array}$$

Exercise: Show that, for any natural number a,

$$\binom{n}{k} = \sum_{j=0}^{a} \binom{a}{j} \binom{n-a}{k-j} .$$

Note that this identity relates each entry in row n of Pascal's triangle to $a+1$ of the entries in row $n-a$. Use this identity to prove the "self-similarity" results above by induction.

Exercise: Show that if p is prime then $1 + x^p + x^{2p} + \cdots + x^{(p-1)p}$ is irreducible, by showing that the substitution $x := 1 + y$ yields a polynomial which is Eisenstein for p.