Estimating central binomial coefficients

Below are some exercises giving successively better estimates on binomial coefficients of the form $\binom{2n}{n}$. The methods are mostly elementary (which here means something like "not using Stirling's approximation"). Hints follow the exercises.

1. Show that, if $m \geq n > 0$ then
$$\left(\frac{m}{n}\right)^n \leq \binom{m}{n} \leq \sum_{k=0}^{n} \binom{m}{k} \leq \left(\frac{em}{n}\right)^n ,$$
and deduce that
$$2^n \leq \binom{2n}{n} \leq (2e)^n .$$

2. By considering a row of Pascal's triangle, show that
$$\frac{4^n}{2n+1} \leq \binom{2n}{n} \leq 4^n .$$

3. Show that
$$\binom{2n}{n} = 4^n \prod_{k=1}^{n} \left(1 - \frac{1}{2k}\right)$$
and thence, by the AM/GM/HM inequality,
$$\frac{4^n}{e\sqrt{2n-1}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{n+1}} .$$

4. Show that
$$\binom{2n}{n} = \frac{4^n}{2n+1} \prod_{k=1}^{n} \left(1 + \frac{1}{2k}\right)$$
and thence
$$\binom{2n}{n} = \frac{4^n}{\sqrt{2n+1}} \sqrt{\prod_{k=1}^{n} \left(1 - \frac{1}{4k^2}\right)} .$$
Use Wallis' product to conclude
$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}} .$$

Hints:

1. For the first inequality, show that if $0 \le k \le n \le m$ then
$$\frac{m}{n} \le \frac{m-k}{n-k},$$
and multiply over k. For the third inequality, show that
$$\binom{m}{k} \le \frac{m^k}{k!} \le \frac{m^n}{n^n} \cdot \frac{n^k}{k!},$$
and sum over k.

2. For a list of positive numbers, the average is at most the maximum, which is at most the sum.

3. First show that
$$\binom{2n}{n-1} = \binom{2n}{n+1} = \frac{n}{n+1}\binom{2n}{n},$$
and deduce
$$\binom{2n}{n} = \left(4 - \frac{2}{n}\right)\binom{2n-2}{n-1}.$$
The formula for $\binom{2n}{n}$ then follows by induction. Applying the AM/GM inequality, as indicated in the exercise, and the standard facts from analysis that
$$\left(1 - \frac{a}{n}\right)^n \le e^{-a} \qquad \text{if } a \le n$$
and
$$\sum_{k=1}^{n} \frac{1}{k} \ge \int_1^{n+1} \frac{dx}{x} = \ln(n+1)$$
yields the upper estimate. The lower estimate follows from the GM/HM inequality in a similar way.

4. The first formula follows by algebraic manipulation of the formula in the previous exercise. Multiplying these two formulæ yields the next.