

A few species of logic

Classical logic

Classical logic

“and”

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

“or”

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

“not”

p	$\neg p$
1	0
0	1

- ▶ like high school algebra, but variables take values 0 and 1 (resp., false and true)
- ▶ operations \wedge , \vee , \neg , \rightarrow instead of $+$, \cdot

“implies”

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Truth tables

Verification that $(q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$ is a theorem:

p	q	r	A $q \rightarrow r$	B $p \rightarrow q$	C $p \rightarrow r$	$B \rightarrow C$	$A \rightarrow B \rightarrow C$
1	1	1	1	1	1	1	1
1	1	0	0	1	0	0	1
1	0	1	1	0	1	1	1
1	0	0	1	0	0	1	1
0	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1
0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1

Axioms and rules for (part of) classical logic

Axioms (all formulas of these forms are free):

1. $A \rightarrow (B \rightarrow A)$
2. $(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$
3. $\neg A \rightarrow (A \rightarrow B)$
4. $\neg\neg A \rightarrow A$

Rule (how to get new formulas):

- ▶ (Modus Ponens) If you have A and $A \rightarrow B$, you can have B .

Example of an axiomatic proof

Ax1. $A \rightarrow (B \rightarrow A)$

Ax2. $(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$

1. $(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$ Ax2
2. $[(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)]$ Ax1
 $\rightarrow (q \rightarrow r) \rightarrow [(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)]$
3. $(q \rightarrow r) \rightarrow [(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)]$ MP (2,3)
4. $[(q \rightarrow r) \rightarrow ((p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r))]$ Ax2
 $\rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow q \rightarrow r)]$
 $\rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)]$
5. $[(q \rightarrow r) \rightarrow (p \rightarrow q \rightarrow r)]$ MP (3,4)
 $\rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)]$
6. $(q \rightarrow r) \rightarrow (p \rightarrow q \rightarrow r)$ Ax1
7. $(q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$ MP (5,6)

What's not to like?

Nonconstructive principles:

- ▶ $p \vee \neg p$
- ▶ $\neg\neg p \rightarrow p$
- ▶ $(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow q)$

Explosion:

- ▶ $p \wedge \neg p \rightarrow q$

Paradoxes of material implication:

- ▶ $p \rightarrow (q \rightarrow p)$
- ▶ $\neg p \rightarrow (p \rightarrow q)$
- ▶ $\neg(p \rightarrow q) \rightarrow p$
- ▶ $(p \wedge q \rightarrow r) \rightarrow (p \rightarrow r) \vee (q \rightarrow r)$
- ▶ $(p \rightarrow q) \wedge (u \rightarrow v) \rightarrow (p \rightarrow v) \vee (u \rightarrow q)$

Modal logic

Modal logic

Modal operators:

$$\left. \begin{array}{l} \Box p \quad \text{"}p \text{ is necessary"} \\ \Diamond p \quad \text{"}p \text{ is possible"} \end{array} \right\} \text{related by } \Box p = \neg \Diamond \neg p$$

Many kinds of necessity:

- ▶ logical
- ▶ physical
- ▶ metaphysical
- ▶ moral
- ▶ practical

Other modalities:

- ▶ p has always been true/will eventually be true
- ▶ p is known/believed/said to be true

Axioms and rules found in modal logics

Often:

- ▶ $\Box(p \rightarrow q) \rightarrow \Box p \rightarrow \Box q$
- ▶ if A is a theorem then $\Box A$ is a theorem

Sometimes:

- ▶ $\Box p \rightarrow \Box \Box p$ (also the dual $\Diamond \Diamond p \rightarrow \Diamond p$)
- ▶ $\Box p \rightarrow p$ (also the dual $p \rightarrow \Diamond p$)
- ▶ $\Diamond \Box p \rightarrow p$ (also the dual $p \rightarrow \Box \Diamond p$)
- ▶ $\Box p \rightarrow \Diamond p$

Rarely:

- ▶ $p \rightarrow \Box p$

Example of a proof in modal logic

Often:

- ▶ $\Box(p \rightarrow q) \rightarrow \Box p \rightarrow \Box q$
 - ▶ if A is a theorem then $\Box A$ is a theorem
-

Theorem: $\Box p \vee \Box q \rightarrow \Box(p \vee q)$

Proof:

$p \rightarrow p \vee q$ is a theorem.

Therefore $\Box(p \rightarrow p \vee q)$ is a theorem.

Therefore $\Box p \rightarrow \Box(p \vee q)$.

Similarly, $\Box q \rightarrow \Box(p \vee q)$.

Therefore $\Box p \vee \Box q \rightarrow \Box(p \vee q)$.

Possible worlds

Classical propositional logic:

- ▶ “interpretation”: choice of truth values for variables p, q, r, \dots
- ▶ “theorem”: formula which is true in all interpretations

“Normal” modal logic:

- ▶ “interpretation”:
 - ▶ collection of worlds, each with truth values for the variables
 - ▶ some worlds can see other worlds (and/or themselves)
 - ▶ “ $\Box p$ ” is true at W if p true at all worlds that W can see
 - ▶ “ $\Diamond p$ ” is true at W if p true at some world that W can see
- ▶ “theorem”: formula true in all worlds in all interpretations

Example of a counterexample using possible worlds

$\Box(p \vee q) \rightarrow \Box p \vee \Box q$ is not a theorem.

Counterexample:

Two worlds, each world seeing itself and the other.

World	p	q	$p \vee q$	$\Box(p \vee q)$	$\Box p$	$\Box q$	$\Box p \vee \Box q$
1	1	0	1	1	0	0	0
2	0	1	1	1	0	0	0

Axioms vs possible worlds

$\Box p \rightarrow \Box \Box p$ “seeing” is transitive

$\Box p \rightarrow p$ “seeing” is reflexive
(every world can see itself)

$\Diamond \Box p \rightarrow p$ “seeing” is symmetric
(if I see you, you can see me)

$\Box p \rightarrow \Diamond p$ every world can see at least one world

Intuitionistic logic

Intuitionistic logic

- ▶ Intuitionism: a philosophy of mathematics
 - ▶ A mathematical statement is “true” when a mathematician makes a mental “construction”.
- ▶ Rejects nonconstructive principles such as
 - ▶ $p \vee \neg p$
 - ▶ $\neg\neg p \rightarrow p$
 - ▶ $(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow q)$
- ▶ Axioms:
 1. $A \rightarrow (B \rightarrow A)$
 2. $(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$
 3. $\neg A \rightarrow (A \rightarrow B)$
 4. $\neg\neg A \rightarrow A$

Asymmetry of negation

$$A \rightarrow \neg\neg A \quad \leftrightarrow \quad \neg\neg\neg\neg A \quad \leftrightarrow \quad \dots$$

$$\neg A \quad \leftrightarrow \quad \neg\neg\neg A \quad \leftrightarrow \quad \neg\neg\neg\neg\neg A \quad \leftrightarrow \quad \dots$$

$$(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p) \quad \checkmark$$

$$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow q) \quad \times$$

$$(p \rightarrow \neg q) \rightarrow (q \rightarrow \neg p) \quad \checkmark$$

$$(\neg p \rightarrow q) \rightarrow (\neg q \rightarrow p) \quad \times$$

$$\neg(p \vee q) \rightarrow \neg p \wedge \neg q \quad \checkmark$$

$$\neg p \wedge \neg q \rightarrow \neg(p \vee q) \quad \checkmark$$

$$\neg(p \wedge q) \rightarrow \neg p \vee \neg q \quad \times$$

$$\neg p \vee \neg q \rightarrow \neg(p \wedge q) \quad \checkmark$$

More asymmetry of negation

$$(p \vee \neg p \rightarrow q) \rightarrow q \quad \times$$

$$(p \vee \neg p \rightarrow \neg q) \rightarrow \neg q \quad \checkmark$$

Sketch of proof:

1. $p \vee \neg p \rightarrow \neg q$ (suppose)
2. $q \rightarrow \neg(p \vee \neg p)$ (contraposition)
3. $q \rightarrow \neg p \wedge \neg \neg p$ (De Morgan)
4. $q \rightarrow \perp$
5. $\neg q$

Corollary:

$$\neg \neg(p \vee \neg p)$$

Grab bag

- ▶ Intuitionistic logic is part of classical logic
(The part you get by omitting double negation, etc.)
- ▶ Classical logic is part of intuitionistic logic
(A is classical theorem iff $\neg\neg A$ is intuitionistic theorem¹)
- ▶ Intuitionistic logic can be treated as a classical modal logic
($\Box p$: “ p is proved”)
- ▶ Intuitionistic logic has a complete topological model
(“truth values” are open sets in the real line)

¹propositional logic only

Multi-valued logic

Reasons to want more than two truth values

- ▶ Maybe some statements are neither true nor false.
 - ▶ future contingents
 - ▶ open conjectures (if “true” means “proved”)
 - ▶ denotation failures
 - ▶ fictional situations
- ▶ Maybe some statements are both true and false.
 - ▶ liar’s paradox
 - ▶ inconsistent information
 - ▶ inconsistent laws
- ▶ Maybe modality can be expressed with extra truth values.
 - ▶ 1: true; 0: false; i : indeterminate
 - ▶ possible: 1 or i
 - ▶ necessary: 1

The three-valued Kleene logic

$p \wedge q$		q		
		1	i	0
p	1	1	i	0
	i	i	i	0
	0	0	0	0

$p \vee q$		q		
		1	i	0
p	1	1	1	1
	i	1	i	i
	0	1	i	0

p	$\neg p$
1	0
i	i
0	1

- ▶ i : “neither true nor false”
- ▶ $p \wedge q$ is true if both p, q true
- ▶ $p \wedge q$ is false if p or q false
- ▶ $p \rightarrow q$ same as $\neg p \vee q$

$p \rightarrow q$		q		
		1	i	0
p	1	1	i	0
	i	1	i	i
	0	1	1	1

Modus ponens in Kleene logic

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$p \wedge (p \rightarrow q) \rightarrow q$
1	1	1	1	1
1	i	i	i	i
1	0	0	0	1
i	1	1	i	1
i	i	i	i	i
i	0	i	i	i
0	1	1	0	1
0	i	1	0	1
0	0	1	0	1

- ▶ $p \wedge (p \rightarrow q) \rightarrow q$ is not a tautology
- ▶ but if p and $p \rightarrow q$ are true, then so is q
(modus ponens is valid)

Deduction theorem

$\vDash A \rightarrow B$	$A \rightarrow B$ is a tautology (true no matter what A, B are)
$A \vDash B$	when A is true, so is B (so, B can be inferred from A)

Equivalent in classical logic, but not in Kleene logic.
Classical logic has a “deduction theorem”.

K_3 has no tautologies at all, not even $p \rightarrow p$.

Some other multi-valued logics

LP (“Logic of Paradox”)

- ▶ same definitions of \neg , \wedge , \vee , \rightarrow as Kleene logic
- ▶ i taken to mean “both true and false”
- ▶ $A \models B$ if when A is true (1 or i), so is B
- ▶ $p \wedge (p \rightarrow q) \rightarrow q$ is a tautology, but modus ponens not valid

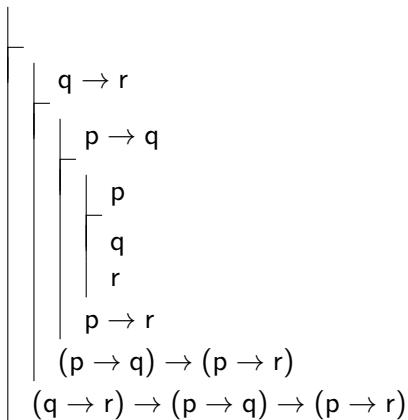
Three-valued Łukasiewicz logic

- ▶ like Kleene logic, except $i \rightarrow i$ has value 1
- ▶ $(p \rightarrow q)$ not the same as $\neg p \vee q$
- ▶ has modus ponens, has contraposition, no excluded middle
- ▶ weird deduction thm: $A \models B$ iff $\models A \rightarrow (A \rightarrow B)$

And lots more. . .

Relevance logic

Subproofs



Suppose A .

Prove B .

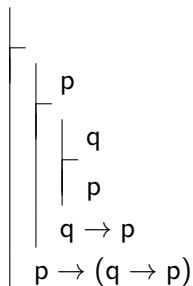
Conclude $A \rightarrow B$.

True statement implied by anything

$$p \rightarrow (q \rightarrow p)$$

(but q not relevant to p !)

Proof in a system with subproofs:



Suppose A .

Prove B .

Conclude $A \rightarrow B$.

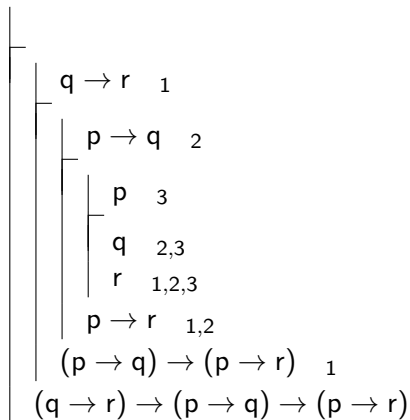
vs

Suppose A .

Prove B using A .

Conclude $A \rightarrow B$.

Track use of assumptions



Suppose A .
Prove B .
Conclude $A \rightarrow B$.

vs

Suppose A .
Prove B using A .
Conclude $A \rightarrow B$.

Axioms for relevant implication

- ▶ $A \rightarrow A$
- ▶ $(A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$
- ▶ $(A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$
- ▶ $(A \rightarrow B \rightarrow C) \rightarrow (B \rightarrow A \rightarrow C)$

Can track use of assumption A with “ $A \rightarrow$ ”!

Grab bag

- ▶ Most popular relevant logic, R , is undecidable.
- ▶ Possible-worlds semantics use a ternary “seeing” relation.
- ▶ No explosion: $p \wedge \neg p \not\rightarrow q$.
- ▶ May distinguish two kinds of \wedge , two kinds of \vee .

Main sources

Books:

- ▶ Priest, *An Introduction to Non-Classical Logic*
- ▶ Gabbay and Guenther eds., *Handbook of Philosophical Logic*
- ▶ Hughes and Cresswell, *An Introduction to Modal Logic*
- ▶ Anderson and Belnap, *Entailment: The Logic of Relevance and Necessity*

Web:

- ▶ Stanford Encyclopedia of Philosophy
- ▶ Wikipedia