A few species of logic

Classical logic

Classical logic

"and"

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

"or"

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

"not"

p	$\neg p$
1	0
0	1

- like high school algebra, but variables take values 0 and 1 (resp., false and true)
- operations $\wedge$, $\vee$, $\neg$, $\rightarrow$ instead of $+$, $\cdot$

"implies"

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Truth tables

Verification that $(q \to r) \to (p \to q) \to (p \to r)$ is a theorem:

			A	B	C		
p	q	r	$q \to r$	$p \to q$	$p \to r$	$B \to C$	$A \to B \to C$
1	1	1	1	1	1	1	1
1	1	0	0	1	0	0	1
1	0	1	1	0	1	1	1
1	0	0	1	0	0	1	1
0	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1
0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1

Axioms and rules for (part of) classical logic

Axioms (all formulas of these forms are free):

1. $A \to (B \to A)$
2. $(A \to (B \to C)) \to (A \to B) \to (A \to C)$
3. $\neg A \to (A \to B)$
4. $\neg\neg A \to A$

Rule (how to get new formulas):

- (Modus Ponens) If you have A and $A \to B$, you can have B.

Example of an axiomatic proof

Ax1. $A \to (B \to A)$
Ax2. $(A \to (B \to C)) \to (A \to B) \to (A \to C)$

1.	$(p \to q \to r) \to (p \to q) \to (p \to r)$	Ax2
2.	$\big[(p \to q \to r) \to (p \to q) \to (p \to r)\big]$ $\to (q \to r) \to \big[(p \to q \to r) \to (p \to q) \to (p \to r)\big]$	Ax1
3.	$(q \to r) \to \big[(p \to q \to r) \to (p \to q) \to (p \to r)\big]$	MP (2,3)
4.	$\big[(q \to r) \to ((p \to q \to r) \to (p \to q) \to (p \to r))\big]$ $\to \big[(q \to r) \to (p \to q \to r)\big]$ $\to \big[(q \to r) \to (p \to q) \to (p \to r)\big]$	Ax2
5.	$\big[(q \to r) \to (p \to q \to r)\big]$ $\to \big[(q \to r) \to (p \to q) \to (p \to r)\big]$	MP (3,4)
6.	$(q \to r) \to (p \to q \to r)$	Ax1
7.	$(q \to r) \to (p \to q) \to (p \to r)$	MP (5,6)

What's not to like?

Nonconstructive principles:

- $p \vee \neg p$
- $\neg\neg p \to p$
- $(\neg q \to \neg p) \to (p \to q)$

Explosion:

- $p \wedge \neg p \to q$

Paradoxes of material implication:

- $p \to (q \to p)$
- $\neg p \to (p \to q)$
- $\neg(p \to q) \to p$
- $(p \wedge q \to r) \to (p \to r) \vee (q \to r)$
- $(p \to q) \wedge (u \to v) \to (p \to v) \vee (u \to q)$

Modal logic

Modal logic

Modal operators:

$$\left.\begin{array}{ll} \Box p & \text{“}p\text{ is necessary”} \\ \Diamond p & \text{“}p\text{ is possible”} \end{array}\right\} \text{related by } \Box p = \neg\Diamond\neg p$$

Many kinds of necessity:

- logical
- physical
- metaphysical
- moral
- practical

Other modalities:

- p has always been true/will eventually be true
- p is known/believed/said to be true

Axioms and rules found in modal logics

Often:

- $\Box(p \to q) \to \Box p \to \Box q$
- if A is a theorem then $\Box A$ is a theorem

Sometimes:

- $\Box p \to \Box\Box p$ (also the dual $\Diamond\Diamond p \to \Diamond p$)
- $\Box p \to p$ (also the dual $p \to \Diamond p$)
- $\Diamond\Box p \to p$ (also the dual $p \to \Box\Diamond p$)
- $\Box p \to \Diamond p$

Rarely:

- $p \to \Box p$

Example of a proof in modal logic

Often:

- $\Box(p \to q) \to \Box p \to \Box q$
- if A is a theorem then $\Box A$ is a theorem

Theorem: $\Box p \vee \Box q \to \Box(p \vee q)$

Proof:
$p \to p \vee q$ is a theorem.
Therefore $\Box(p \to p \vee q)$ is a theorem.
Therefore $\Box p \to \Box(p \vee q)$.
Similarly, $\Box q \to \Box(p \vee q)$.
Therefore $\Box p \vee \Box q \to \Box(p \vee q)$.

Possible worlds

Classical propositional logic:

- "interpretation": choice of truth values for variables $p, q, r, \ldots$
- "theorem": formula which is true in all interpretations

"Normal" modal logic:

- "interpretation":
  - collection of worlds, each with truth values for the variables
  - some worlds can see other worlds (and/or themselves)
  - "$\Box p$" is true at W if p true at all worlds that W can see
  - "$\Diamond p$" is true at W if p true at some world that W can see
- "theorem": formula true in all worlds in all interpretations

Example of a counterexample using possible worlds

$\Box(p \vee q) \rightarrow \Box p \vee \Box q$ is not a theorem.

Counterexample:

Two worlds, each world seeing itself and the other.

World	p	q	$p \vee q$	$\Box(p \vee q)$	$\Box p$	$\Box q$	$\Box p \vee \Box q$
1	1	0	1	1	0	0	0
2	0	1	1	1	0	0	0

Axioms vs possible worlds

$\Box p \to \Box\Box p$	"seeing" is transitive
$\Box p \to p$	"seeing" is reflexive (every world can see itself)
$\Diamond\Box p \to p$	"seeing" is symmetric (if I see you, you can see me)
$\Box p \to \Diamond p$	every world can see at least one world

Intuitionistic logic

Intuitionistic logic

- Intuitionism: a philosophy of mathematics
  - A mathematical statement is “true” when a mathematician makes a mental “construction”.
- Rejects nonconstructive principles such as
  - $p \vee \neg p$
  - $\neg\neg p \to p$
  - $(\neg q \to \neg p) \to (p \to q)$
- Axioms:
  1. $A \to (B \to A)$
  2. $(A \to (B \to C)) \to (A \to B) \to (A \to C)$
  3. $\neg A \to (A \to B)$
  4. $\cancel{\neg\neg A \to A}$

Asymmetry of negation

$$A \quad \rightarrow \quad \neg\neg A \quad \leftrightarrow \quad \neg\neg\neg\neg A \quad \leftrightarrow \quad \cdots$$

$$\neg A \quad \leftrightarrow \quad \neg\neg\neg A \quad \leftrightarrow \quad \neg\neg\neg\neg\neg A \quad \leftrightarrow \quad \cdots$$

$(p \to q) \to (\neg q \to \neg p)$	$\checkmark$	$\neg(p \vee q) \to \neg p \wedge \neg q$	$\checkmark$
$(\neg q \to \neg p) \to (p \to q)$	$\times$	$\neg p \wedge \neg q \to \neg(p \vee q)$	$\checkmark$
$(p \to \neg q) \to (q \to \neg p)$	$\checkmark$	$\neg(p \wedge q) \to \neg p \vee \neg q$	$\times$
$(\neg p \to q) \to (\neg q \to p)$	$\times$	$\neg p \vee \neg q \to \neg(p \wedge q)$	$\checkmark$

More asymmetry of negation

$$(p \vee \neg p \to q) \to q \qquad \times$$
$$(p \vee \neg p \to \neg q) \to \neg q \qquad \checkmark$$

Sketch of proof:

1. $p \vee \neg p \to \neg q$ (suppose)
2. $q \to \neg(p \vee \neg p)$ (contraposition)
3. $q \to \neg p \wedge \neg\neg p$ (De Morgan)
4. $q \to \bot$
5. $\neg q$

Corollary:
$\neg\neg(p \vee \neg p)$

Grab bag

- Intuitionistic logic is part of classical logic
  (The part you get by omitting double negation, etc.)
- Classical logic is part of intuitionistic logic
  (A is classical theorem iff $\neg\neg A$ is intuitionistic theorem[1])
- Intuitionistic logic can be treated as a classical modal logic
  ($\Box p$: "p is proved")
- Intuitionistic logic has a complete topological model
  ("truth values" are open sets in the real line)

[1] propositional logic only

Multi-valued logic

Reasons to want more than two truth values

- Maybe some statements are neither true nor false.
  - future contingents
  - open conjectures (if "true" means "proved")
  - denotation failures
  - fictional situations
- Maybe some statements are both true and false.
  - liar's paradox
  - inconsistent information
  - inconsistent laws
- Maybe modality can be expressed with extra truth values.
  - 1: true; 0: false; i: indeterminate
  - possible: 1 or i
  - necessary: 1

The three-valued Kleene logic

$p \wedge q$		q		
		1	i	0
	1	1	i	0
p	i	i	i	0
	0	0	0	0

$p \vee q$		q		
		1	i	0
	1	1	1	1
p	i	1	i	i
	0	1	i	0

p	$\neg p$
1	0
i	i
0	1

- i: "neither true nor false"
- $p \wedge q$ is true if both p, q true
- $p \wedge q$ is false if p or q false
- $p \to q$ same as $\neg p \vee q$

$p \to q$		q		
		1	i	0
	1	1	i	0
p	i	1	i	i
	0	1	1	1

Modus ponens in Kleene logic

p	q	$p \to q$	$p \wedge (p \to q)$	$p \wedge (p \to q) \to q$
1	1	1	1	1
1	i	i	i	i
1	0	0	0	1
i	1	1	i	1
i	i	i	i	i
i	0	i	i	i
0	1	1	0	1
0	i	1	0	1
0	0	1	0	1

- $p \wedge (p \to q) \to q$ is not a tautology
- but if p and $p \to q$ are true, then so is q (modus ponens is valid)

Deduction theorem

$\models A \to B$	$A \to B$ is a tautology (true no matter what A, B are)
$A \models B$	when A is true, so is B (so, B can be inferred from A)

Equivalent in classical logic, but not in Kleene logic.
Classical logic has a "deduction theorem".

K_3 has no tautologies at all, not even $p \to p$.

Some other multi-valued logics

LP ("Logic of Paradox")

- same definitions of $\neg$, $\wedge$, $\vee$, $\rightarrow$ as Kleene logic
- i taken to mean "both true and false"
- $A \vDash B$ if when A is true (1 or i), so is B
- $p \wedge (p \rightarrow q) \rightarrow q$ is a tautology, but modus ponens not valid

Three-valued Łukasiewicz logic

- like Kleene logic, except $i \rightarrow i$ has value 1
- ($p \rightarrow q$ not the same as $\neg p \vee q$)
- has modus ponens, has contraposition, no excluded middle
- weird deduction thm: $A \vDash B$ iff $\vDash A \rightarrow (A \rightarrow B)$

And lots more...

Relevance logic

Subproofs

$$
\begin{array}{l|l}
 & \\ \hline
 & \begin{array}{l|l}
 & q \to r \\ \hline
 & \begin{array}{l|l}
 & p \to q \\ \hline
 & \begin{array}{l|l}
 & p \\ \hline
 & q \\
 & r
 \end{array} \\
 & p \to r
 \end{array} \\
 & (p \to q) \to (p \to r)
 \end{array} \\
 & (q \to r) \to (p \to q) \to (p \to r)
\end{array}
$$

Suppose A.
Prove B.
Conclude $A \to B$.

True statement implied by anything

$p \to (q \to p)$ (but q not relevant to p!)

Proof in a system with subproofs:

$$\begin{array}{l}
\quad p \\
\quad\quad q \\
\quad\quad p \\
\quad q \to p \\
p \to (q \to p)
\end{array}$$

Suppose A.
Prove B.
Conclude $A \to B$.

vs

Suppose A.
Prove B using A.
Conclude $A \to B$.

Track use of assumptions

$$
\begin{array}{llll}
 & q \to r & & 1 \\
 & \quad p \to q & & 2 \\
 & \quad\quad p & & 3 \\
 & \quad\quad q & & 2,3 \\
 & \quad\quad r & & 1,2,3 \\
 & \quad p \to r & & 1,2 \\
 & (p \to q) \to (p \to r) & & 1 \\
(q \to r) \to (p \to q) \to (p \to r) & & &
\end{array}
$$

Suppose A.
Prove B.
Conclude $A \to B$.

vs

Suppose A.
Prove B using A.
Conclude $A \to B$.

Axioms for relevant implication

- $A \to A$
- $(A \to B \to C) \to (A \to B) \to (A \to C)$
- $(A \to B) \to (B \to C) \to (A \to C)$
- $(A \to B \to C) \to (B \to A \to C)$

Can track use of assumption A with "$A \to$"!

Grab bag

- Most popular relevant logic, R, is undecidable.
- Possible-worlds semantics use a ternary “seeing” relation.
- No explosion: $p \wedge \neg p \not\to q$.
- May distinguish two kinds of $\wedge$, two kinds of $\vee$.

Main sources

Books:

- Priest, *An Introduction to Non-Classical Logic*
- Gabbay and Guenthner eds., *Handbook of Philosophical Logic*
- Hughes and Cresswell, *An Introduction to Modal Logic*
- Anderson and Belnap, *Entailment: The Logic of Relevance and Necessity*

Web:

- Stanford Encyclopedia of Philosophy
- Wikipedia